PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Effect of Global-Scale, Steady-State Convection and Elastic-Gravitational Asphericities on Helioseismic Oscillations

Eugene M. Lavely and Michael H. Ritzwoller

Phil. Trans. R. Soc. Lond. A 1992 **339**, 431-496
doi: 10.1098/rsta.1992.0048

The effect of global-scale, steady-state convection and elastic-gravitational asphericities on helioseismic oscillations

By Eugene M. Lavely[1] and Michael H. Ritzwoller[2]

[1] *Advanced Study Program and High Altitude Observatory, National Center for Atmospheric Research, P.O. Box 3000, Boulder, Colorado 80307-3000, U.S.A.*

[2] *Department of Physics, University of Colorado, Campus Box 390, Boulder, Colorado 80309-0390, U.S.A.*

Contents

Phil. Trans. R. Soc. Lond. A (1992) **339**, 431–496

Printed in Great Britain

18

Vol. 339. A (15 June 1992)

In this paper we derive a theory, based on quasi-degenerate perturbation theory, that governs the effect of global-scale, steady-state convection and associated static asphericities in the elastic-gravitational variables (adiabatic bulk modulus κ, density ρ, and gravitational potential ϕ) on helioseismic eigenfrequencies and eigenfunctions and present a formalism with which this theory can be applied computationally. The theory rests on three formal assumptions: (1) that convection is temporally steady in a frame corotating with the Sun, (2) that accurate eigenfrequencies and eigenfunctions can be determined by retaining terms in the seismically perturbed equations of motion only to first order in p-mode displacement, and (3) that we are justified in retaining terms only to first order in convective velocity (this is tantamount to assuming that the convective flow is anelastic). The most physically unrealistic assumption is (1), and we view the results of this paper as the first step toward a more general theory governing the seismic effects of time-varying fields. Although the theory does not govern the seismic effects of non-stationary flows, it can be used to approximate the effects of unsteady flows on the acoustic wavefield if the flow is varying smoothly in time. The theory does not attempt to model seismic modal amplitudes since these are governed, in part, by the exchange of energy between convection and acoustic motions which is not a part of this theory. However, we show how theoretical wavefields can be computed given a description of the stress field produced by a source process such as turbulent convection.

The basic reference model that will be perturbed by rotation, convection, structural asphericities, and acoustic oscillations is a spherically symmetric, non-rotating, non-magnetic, isotropic, static solar model that, when subject to acoustic oscillations, oscillates adiabatically. We call this the SNRNMAIS model. An acoustic mode of the SNRNMAIS model is denoted by $k = (n, l, m)$, where n is the radial order, l is the harmonic degree, and m is the azimuthal order of the mode.

The main result of the paper is the general matrix element $H^{m'm}_{n'n,l'l}$ for steady-state convection satisfying the anelastic condition with static structural asphericities. It is written in terms of the radial, scalar eigenfunctions of the SNRNMAIS model, resulting in equations (90)–(110). We prove Rayleigh's principle in our derivation of quasi-degenerate perturbation theory which, as a by-product, yields the general matrix element. Within this perturbative method, modes need not be exactly degenerate in the SNRNMAIS solar model to couple, only nearly so. General matrix elements compose the hermitian supermatrix $\boldsymbol{Z}$. The eigenvalues of the supermatrix are the eigenfrequency perturbations of the convecting, aspherical model and the eigenvector components of $\boldsymbol{Z}$ are the expansion coefficients in the linear combination forming the eigenfunctions in which the eigenfunctions of the SNRNMAIS solar model act as basis functions.

The properties of the Wigner $3j$ symbols and the reduced matrix elements composing $H^{m'm}_{n'n,l'l}$ produce selection rules governing the coupling of SNRNMAIS modes that hold even for time-varying flows. We state selection rules for both quasi-degenerate and degenerate perturbation theories. For example, within degenerate perturbation theory, only odd-degree s toroidal flows and even degree structural asphericities, both with $s \leqslant 2l$, will couple and/or split acoustic modes with harmonic degree l. In addition, the frequency perturbations caused by a toroidal flow display odd symmetry with respect to the degenerate frequency when ordered from the minimum to the maximum frequency perturbation.

We consider the special case of differential rotation, the odd-degree, axisymmetric, toroidal component of general convection, and present the general matrix element and selection rules under quasi-degenerate perturbation theory. We argue that due to the spacing of modes that satisfy the selection rules, quasi-degenerate coupling can, for all practical purposes, be neglected in modelling the effect of low-degree differential rotation on helioseismic data. In effect, modes that can couple through low-degree differential rotation are too far separated in frequency to couple strongly. This is not the case for non-axisymmetric flows and asphericities where near degeneracies will regularly occur, and couplings can be relatively strong especially among SNRNMAIS modes within the same multiplet.

All derivations are performed and all solutions are presented in a frame corotating with the mean solar angular rotation rate. Equation (18) shows how to transform the eigenfrequencies and eigenfunctions in the corotating frame into an inertial frame. The transformation has the effect that each eigenfunction in the inertial frame is itself time varying. That is, a mode of oscillation, which is defined to have a single frequency in the corotating frame, becomes multiply periodic in the inertial frame.

---

1. Introduction

Helioseismic images of the acoustic velocity field of the Sun are providing new and unique information about solar structure and dynamics. To continue to utilize effectively the information provided by the continually improving data-sets will require a thorough understanding of the way in which solar structures and processes affect helioseismic data. It is upon such an understanding of these forward problems that any future inversions will rest.

We consider here the helioseismic effect of one such solar process: convection. The purpose of this paper is to present a theory that governs the effect of large-scale, steady-state convection, with associated asphericities in the structural elastic-gravitational variables (adiabatic bulk modulus κ, density ρ, and gravitational potential ϕ), on helioseismic oscillations. Many studies have modelled the seismic effect of differential rotation, the long-wavelength axisymmetric component of convection (Duvall & Harvey 1984; Brown 1985; Duvall *et al.* 1986; Libbrecht 1986, 1989; Brown & Morrow 1987; Brown *et al.* 1989; Rhodes *et al.* 1990; Thompson 1990; Ritzwoller & Lavely 1991). However, to data, studies of the seismic effect of non-axisymmetric convection are rather sparse. In an asymptotic treatment, Gough & Toomre (1983) calculated the frequency shift of an acoustic mode due to advection by a purely horizontal flow. Brown (1984) calculated the influence of turbulent convection on modal degenerate frequencies. The scattering of sound by an isolated, steady laminar compact vortex was considered by Bogdan (1989). Hill *et al.* (1983) and Hill (1988, 1989) have applied ray-theoretic methods to infer horizontal convective velocities near the solar surface by using frequency shifts of dispersion curves. All of these studies make restrictive assumptions about the geometry of the flow field including either that the flows are horizontal in a plane-parallel medium or demonstrate cylindrical symmetry, and none attempts to model wave-front distortion and deflection caused by convection. In summary, to the best of our knowledge no general theory for the effect of convection on global helioseismic oscillations currently exists.

The theory presented in this paper differs from previous studies in the following ways. (1) Our theory is non-asymptotic. In principle, the results are accurate for all wavelengths and frequencies of helioseismic oscillation to the extent that the assumptions of the theory are valid. (2) It is derived within a spherical geometry. Previous investigations that modelled convective effects within a non-spherical geometry are appropriate for short-wavelength convection but inappropriate for global-scale convection which is the subject of this paper. (3) The theory presented here makes no assumptions about the geometry of the flow. We represent general non-axisymmetric flow fields comprising both poloidal and toroidal components in terms of vector spherical harmonics, which are complete basis functions for a vector field in a sphere. (4) Our approach is modal-theoretic rather than ray-theoretic. From a travelling wave perspective this means that both wave-front deformation as well as the perturbation in local sound speed by convection are modelled. In modal-theoretic language, convection results in modal coupling as well as splitting.

In Lavely & Ritzwoller (1992) we implement the theory presented in this paper using a numerical simulation of large-scale convection and discuss the observational consequences of the theory. In particular, we show that the helioseismic frequencies, displacement patterns, and line-widths of an aspherical solar model are appreciably altered relative to the corresponding quantities calculated from a model with differential rotation alone.

(*a*) *Modal notation and terminology*

The basic reference model to which all subsequent structural perturbations and processes will be added is a solar model that is spherically symmetric, non-rotating, non-magnetic, isotropic, and static, subject to adiabatic acoustic oscillations. We refer to this as the SNRNMAIS solar model. An acoustic mode of oscillation of any solar model is defined to be a characteristic spatial displacement pattern that oscillates with a single frequency.

An acoustic mode of a SNRNMAIS model is uniquely identified by a single triple of quantum numbers (n, l, m) that denote, respectively, the radial order, harmonic degree, and azimuthal order of the mode. A modal frequency for such a model is simply the degenerate frequency of the multiplet ${}_nS_l$ that comprises the $(2l+1)$ modes with identical n and l values. Any symmetry-breaking agent such as rotation, magnetic fields, or convection will lift this $(2l+1)$ degeneracy and split the frequencies of the modes composing the multiplet. We call any model with such a symmetry-breaking perturbation a non-SNRNMAIS model. A major goal of this paper is to provide formulae with which to calculate the modal eigenfunctions and eigenfrequencies of both SNRNMAIS and non-SNRNMAIS solar models. The perturbations of the non-SNRNMAIS solar model will be assumed to have small magnitude and be stationary in a frame corotating with the Sun. If the symmetry-breaking agent is axisymmetric, as is differential rotation, then to a good approximation the spatial structure of each mode will remain specified by the same triplet of quantum numbers. For a general, non-axisymmetric perturbation such as a convective flow field, the eigenfunction (or spatial displacement pattern) of each mode is a linear combination of the eigenfunctions of the SNRNMAIS solar model. We call this phenomenon oscillation–oscillation coupling or interaction to distinguish it from oscillation–convection coupling, the exchange of energy between acoustic modes and convective motions. The acoustic modes that are said to couple as a result of a convective flow or a structural asphericity are SNRNMAIS modes. The modes of the non-SNRNMAIS solar model do not couple.

Two distinct modes of the SNRNMAIS solar model are orthogonal (in the sense of (4)) and are said to be isolated from one another. These modes may couple when the reference model is perturbed either by a structural perturbation or a convective flow. A multiplet composed of modes whose combined eigenspace is orthogonal to the combined eigenspace of the modes composing all other multiplets is said to be isolated or self-coupled. The degree of coupling between SNRNMAIS modes is a function of several factors, among which are the strength of the asphericity or convective flow producing the coupling, the proximity of the eigenfrequencies of the modes, the relation between the geometries of the perturbation and the oscillations which is encoded in a set of analytical angular selection rules, and the similarity of

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

the radial eigenfunctions of the two modes. When two SNRNMAIS modes $k = (n, l, m)$ and $k' = (n', l', m')$ couple, the strength of interaction is described by the general matrix element $H^{m'm}_{n'n,l'l}$. The matrix $\boldsymbol{H}_{n'n,l'l}$ composed of all the general matrix elements for the multiplets ${}_nS_l$ and ${}_{n'}S_{l'}$ is of dimension $(2l'+1)\times(2l+1)$ and is called the general matrix. The square general matrix $\boldsymbol{H}_{nn,ll}$ is called the splitting matrix and governs self-coupling. The eigenfrequencies of non-isolated modes that couple within or across n or l are the eigenvalues of an assemblage of block diagonal splitting matrices and off-block diagonal general matrices. The entire assemblage is called the supermatrix $\boldsymbol{Z}$.

Since the acoustic modes of the SNRNMAIS solar model are spheroidal, their eigenfunctions $\boldsymbol{s}_k(\boldsymbol{r})$ may be written in the form

$$\boldsymbol{s}_k(\boldsymbol{r}) = {}_nU_l(r)\, Y^m_l(\theta,\phi)\,\hat{\boldsymbol{r}} + {}_nV_l(r)\,\nabla_1\, Y^m_l(\theta,\phi), \tag{1}$$

where ${}_nU_l(r)$ and ${}_nV_l(r)$ are the scalar radial eigenfunctions for harmonic degree l and radial order n. The coordinates (r, θ, ϕ) are spherical polar coordinates (where θ is colatitude) and $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\varphi}}$ denote unit vectors in the coordinate directions. The gravitational potential scalar eigenfunction, $\delta_n\,\phi_l(r)$, and its radial derivative $\delta_n\,\dot{\phi}_l(r)$, ${}_nU_l(r)$, and ${}_nV_l(r)$ form the set of scalar radial eigenfunctions. The surface gradient operator is given by

$$\begin{aligned}\nabla_1 &= r(\nabla - \hat{\boldsymbol{r}}(\hat{\boldsymbol{r}}\cdot\nabla)) \\ &= \hat{\boldsymbol{\theta}}\frac{\partial}{\partial\theta} + \frac{\hat{\boldsymbol{\varphi}}}{\sin\theta}\frac{\partial}{\partial\phi}.\end{aligned} \tag{2}$$

The function $Y^m_l(\theta,\phi)$ is a spherical harmonic of degree l and azimuthal order m defined using the convention of Edmonds (1960):

$$\int_0^{2\pi}\int_0^{\pi} [Y^{m'}_{l'}(\theta,\phi)]^* Y^m_l(\theta,\phi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \delta_{m'm}\,\delta_{l'l}, \tag{3}$$

where integration is over the unit sphere and * denotes the complex conjugate. Henceforth, we drop the subscripts n and l in equation (1) and use instead $U = {}_nU_l(r)$, $U' = {}_{n'}U_{l'}(r)$ and so on. The SNRNMAIS spatial vector eigenfunctions satisfy an orthogonality condition given by

$$\int \rho_0\,\boldsymbol{s}^*_{k'}\cdot\boldsymbol{s}_k\,\mathrm{d}^3\boldsymbol{r} = N_k\,\delta_{k'k}, \tag{4}$$

where
$$N_k = \int_0^{R_\odot} \rho_0[UU' + l(l+1)\,VV']\,r^2\,\mathrm{d}r, \tag{5}$$

and $\mathrm{d}^3\boldsymbol{r} = r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}r$. Henceforth, an integral sign without limits, as in equation (4), will denote a three-dimensional integration over the volume of the solar model.

Perturbation theoretic techniques are usually used to calculate split acoustic mode eigenfrequencies and eigenfunctions of a perturbed model. We show how quasi-degenerate perturbation theory can be applied to determine these quantities for a non-SNRNMAIS solar model. We use the eigenfunctions of the SNRNMAIS model as basis functions to represent the mode $\tilde{\boldsymbol{s}}_j(\boldsymbol{r},t)$ where j is a mode index:

$$\tilde{\boldsymbol{s}}_j(\boldsymbol{r},t) = \left[\sum_{k\in K} a^j_k\,\boldsymbol{s}_k(\boldsymbol{r})\right]\mathrm{e}^{\mathrm{i}\omega_j t}. \tag{6}$$

We show how to determine the appropriate eigenspace K required to represent the mode, derive the expansion coefficient a_k^j for each component of the eigenspace, and derive expressions for the split eigenfrequency of the mode $\omega_j = \omega_{\text{ref}} + \delta\omega_j$ where ω_{ref} denotes a reference frequency. Note that the expansion in equation (6) excludes toroidal modes and therefore does not represent a complete set. In the case of the Sun the toroidal modes are all of zero or near zero frequency. The central tenant of quasi-degenerate perturbation theory is that only modes that are very similar in frequency will significantly interact. Since the p-modes we are considering typically have frequencies of 1.5 mHz or greater, we conclude that toroidal modes are of no significance and that inclusion of spheroidal terms alone in equation (6) is adequate.

The major theoretical result of the paper is analytical expressions for the general matrix elements that compose the supermatrix (or splitting matrix in the case of self-coupling). The perturbed modal frequency $\delta\omega_j$ is simply an eigenvalue of the supermatrix (or splitting matrix), and the expansion coefficients are simply the eigenvector components a_k^j. We do not attempt to present a theory that accurately predicts modal amplitudes, but only modal eigenfrequencies and eigenfunctions; the formal assumptions of the theory discussed in §1*b* will reflect this point.

(*b*) *Assumptions and their implications*

Although the theory presented in this paper is more general than previous work, its application is restricted both by practical considerations and by the set of assumptions upon which it is formally based. The major practical limitation is that the convective structures considered should be global in extent. For example, although it is possible to represent a single small-scale convective vortex in terms of vector spherical harmonics, there are better representations and doing so would probably be a misuse of this theory. Thus, though the theory holds for all but very short wavelength, turbulent convection, it will be most usefully applied to long wavelength flows. There is a caveat: spatially repetitive small-scale structures, such as the solar granulation, can be well represented by vector spherical harmonics and are not beyond the practical limitations of this theory. The theory applies to both p-modes and g-modes but since the latter have not been observed unambiguously, our discussion will centre on p-modes.

Much more restrictive are the following set of formal assumptions. (1) The convection is steady in time. As we will discuss in §2, this assumption is necessary for the equations of motion to separate. The asphericities in the structural elastic-gravitational variables will also be assumed to be time invariant. (2) We retain terms in the seismically perturbed equations of motion only to first order in p-mode displacement. Thus we derive and use linearized equations of motion. (3) We also retain terms in the seismically perturbed equations of motion only to first order in convective velocity. This is done so that acoustic oscillations and convection do not exchange energy and to this extent can be considered independently. This is tantamount to the requirement that the convective flow field is anelastic. We discuss briefly the implications of each of these assumptions in turn. Arguments are presented to justify assumptions (2) and (3) in §1*e*.

Formal assumption (1). The convection is steady. If convection is steady in time, each identically directed acoustic wave that propagates through a given region will experience the same convective effect. Multiply orbiting waves propagating along near great circles will experience a constructively accumulating effect in that region. In this case, the split modal frequency associated with the propagating wave will be time invariant. If the convective state changes appreciably during the time it takes an acoustic wave to execute a single orbit, then the convective effect will vary between orbits. In fact, the effect may destructively accumulate. Consequently, modal frequencies would be time varying, leading to an effective line-broadening of modal resonance functions. This line-broadening is not a part of the theory presented in this paper and the seismic effect of aspects of convection that are rapidly evolving in time cannot be determined from the results presented here. Of particular significance is the fact that the effect of the shearing of sectoral or banana cell modes of convection by differential rotation cannot be modelled within this theory. Rather, the results in this paper represent the first steps toward constructing a more general theory that governs time-varying fields.

Although the results in this paper are correctly applied only to steady-state convection, they may be most useful if seen to provide instantaneous frequencies and displacement patterns for a time-varying convective field. These instantaneous frequencies would be accurate over the lifetime of the convection cell which, for long-lived modes of convection, may be appreciable. In this case, the steady-state assumption would amount to a short-time approximation. For example, since the shearing of convective patterns takes time to develop, the results presented here are applicable until the shearing effects accumulate. The numerical simulations of Glatzmaier & Gilman (1981, 1982) show that some components of flow have lifetimes on the order of weeks. Furthermore, there are certain observable solar features, in particular active longitudes and coronal holes, that appear to evolve relatively unsheared by differential rotation. If these features are somehow anchored at depth in convective structures, then their existence is further evidence for a relatively stable component of flow deep in the convection zone.

From the view of the steady-state assumption as a short-time approximation, it is straightforward to implement a numerical formalism to approximate the time-varying acoustic wavefield if we assume that the variations in convection are temporally smooth. We would calculate a time sequence of instantaneously valid eigenfrequencies and eigenvectors on a coarse set of time knots where at each knot the flow field is assumed to be stationary. We would then interpolate the eigenfrequencies and eigenvectors onto a finer time grid and allow the wavefield to evolve continuously through each of the intervals between the knots.

Formal assumption (*2*). The equations of motion are linearized. Neglecting higher-order terms than first in the seismically perturbed quantities amounts to neglecting seismic self-advective effects. In particular, the self-advection of the displacement field is neglected which is tantamount to assuming that the total acoustic displacement in a region is much smaller than the displacements produced by convection during the passage of a wave. As we discuss in §§1*c* and *e*, the accuracy of this assumption improves with depth. The application of the theory will be most accurate for acoustic paths below the strongly super-adiabatic layer near the solar surface where turbulence is most vigorous.

Formal assumption (*3*). Reynolds stresses terms and terms second order in the convective velocity such as the self-advective term are discarded. Discarding the latter amounts to assuming that convective velocities are relatively small. Ignoring Reynolds stresses, which are proportional to the laplacian of the convective velocity, is equivalent to neglecting turbulent viscosity and requires that convective wavelengths be relatively large. This implies that convection–oscillation coupling is neglected so that there is no mechanism by which convection and the acoustic oscillations can exchange energy. In particular, we assume that convective flows do not generate acoustic waves and, therefore, we require that the flows satisfy the unperturbed continuity equation commonly called the anelastic condition. This condition eliminates potential sources, sinks, and cavitation in the flow field. Thus we view convection as a sort of passive background on which acoustic oscillations are superposed. It deforms acoustic wave-fronts and perturbs local sound speeds, but does not exchange energy with acoustic waves. The validity of these assumptions is poor near the surface but, as with formal assumption (2), improves with depth below the photosphere.

In summary, the implications of these assumptions are that the convective fields to which the theory is applicable should be of relatively long wavelength, steady in time or at least relatively long-lived, and well below the photosphere. Giant-cell convection satisfies these criteria and provides the best target for the application of the theory presented herein. In the remainder of this section we discuss solar convection, review the evidence for the existence of giant-cell convection, and attempt to justify the use of linearized equations of motion to determine the seismic effect of giant-cells.

(*c*) *Solar convection and its seismic effects*

Observation of the distinct cellular motions of granules and supergranules suggests that there are preferred scales of motion for thermal convection. The common picture of convection is that the Sun contains a multiplicity of scales of motion ranging from the Kolmogorov microscales at the short end to differential rotation which is global in extent. At intermediate length scales, convective motions are thought to be organized into granules, supergranules, and giant-cells. Temporal scales also range from a few minutes for granule overturn times to weeks for the largest scale of giant-cells deep in the convection zone. Goldreich & Kumar (1988) present a recent review of turbulence. Bray *et al.* (1984) and Gilman (1987) provide overviews of the physics and morphology of granules, supergranules, and giant-cells. For a recent review of solar convection, see Spruit *et al.* (1990).

To discuss qualitatively the likely general characteristics of convection below the photosphere, we look to mixing-length theory for guidance. In the mixing-length picture of convection one would take the mixing length, the Mach number, and the velocity and timescales of convection to be given, respectively, by

$$H \sim \alpha H_{\mathrm{p}}, \tag{7}$$

$$M \sim [gF_{\mathrm{c}} HQ\rho^{\frac{1}{2}}/4\kappa^{\frac{3}{2}}c_p T]^{\frac{1}{3}}, \tag{8}$$

$$v_H \sim cM, \tag{9}$$

$$\tau_H \sim H/v_H, \tag{10}$$

where $H_{\mathrm{p}} = P/(\rho g)$ is the pressure scale height, α is the ratio of the mixing length to H_{p}, P is pressure, T is temperature, g is the gravity, c_p is the specific heat at constant pressure, and c is the sound speed. We have set $Q = (4-3\beta)\,\beta^{-1}$ where β is the ratio of the gas pressure to the total pressure. The convective flux can be calculated by using

$$F_{\mathrm{c}} = \frac{L_{\odot}}{4\pi r^2}\left[\frac{\nabla_{\mathrm{r}}-\nabla}{\nabla_{\mathrm{r}}}\right], \tag{11}$$

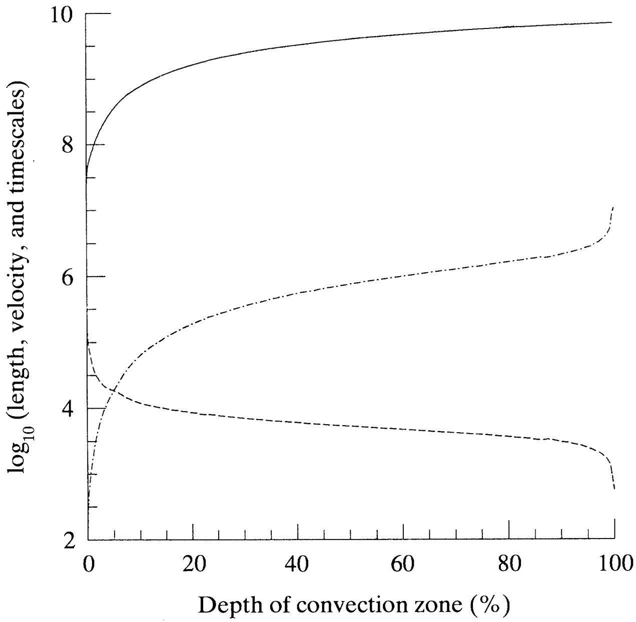

Figure 1. Characteristic length, velocity, and time scales of convective eddies as predicted by mixing length theory (equations (7), (9) and (10)) plotted as a function of depth. The solar model of Podsiadlowski (1989) was used to calculate these quantities. ——, length scale (cm); -—-—-—-, timescale (s); -----, velocity scale (cm s^{-1}).

where $L_{\odot}$ is the solar luminosity, ∇ is the temperature gradient (using standard notation), and ∇_r is the radiative temperature gradient. We have used equation (14.58) of Cox & Giuli (1968) to obtain equation (8). Figure 1 is a plot of the characteristic length, velocity, and timescales of convection predicted by equations (7), (9), and (10) by using the solar model of Podsiadlowski (1989) with α taken to be 1.305. The predicted time and velocity scales near the surface correspond well with observations of solar granulation.

Convection at all depths in the convection zone will affect helioseismic oscillations. The p-mode horizontal wavelengths range in size from the smallest to the largest scales of the convective motions and the dominant modal frequencies coincide with the characteristic overturn times for convective motions near the surface. Since the energy and the characteristic length and timescales of convection vary with depth, the physics of interaction between acoustic modes and convection will necessarily also vary. For example, granule and sub-granule scale motions are thought to be the source of the acoustic oscillations (Goldreich & Kumar 1988). To model the total effect of convection at this scale on the acoustic oscillations would be very difficult as it would involve modelling convection–oscillation coupling in addition to oscillation–oscillation coupling. As the formal assumptions indicate, we have set for ourselves a simpler task: to model the effect of deeper, long-wavelength convection such as giant-cells that, we argue in §1e, exchange very little energy with acoustic oscillations.

Though, as figure 1 shows, it is likely that the characteristic temporal and spatial scales of convection vary continuously across the convection zone, convective processes can be segregated into two concentric shells (an outer shell and an inner shell), with convection in each shell being dominated by distinct processes. In §1*e* we attempt to quantify the extent of the outer shell; here we discuss the characteristics of the convection and its seismic effects in each shell.

The outer shell occupies the top few pressure scale heights where the acoustic and convective physics are most complex. Convection in this shell is highly turbulent and displays relatively short characteristic lifetimes and length scales. The convective velocity in the outer shell is an appreciable fraction of the local sound speed ($M \approx 0.3$), the timescales of the turbulence and of the acoustic radiation are commensurable, and the amplitudes of the *p*-modes and the convective flows are largest. Goldreich & Keely (1977*a*, *b*) and Goldreich & Kumar (1988, 1990) calculated *p*-mode energies under the assumption they are excited by turbulent convection. Their work shows that acoustic wave emission and absorption in the Sun principally take place through interaction with turbulence in the top few scale heights of the convection zone. We define the radial extent of the outer shell as the region of significant interaction between the *p*-modes and convection. We argue in §1*e* that this region is also where three-mode coupling is most appreciable.

In the outer shell, convective cells evolve rapidly (Stein & Nordlund 1989; Title *et al.* 1989). If, in addition, cells are distributed isotropically in space, then they will produce little accumulated splitting effect on globally propagating waves. There will be local acoustic effects, but the isotropic assumption guarantees that the net global effect on frequency will be small. However, acoustic modal amplitudes, damping rates, and degenerate frequencies will be affected by outer shell processes (Brown 1984; Christensen-Dalsgaard & Fransden 1983; Christensen-Dalsgaard *et al.* 1989; Kumar & Goldreich 1989) such as convection–oscillation coupling, three-mode coupling, and radiative damping.

The inner shell is much larger than the outer shell and lies directly beneath it, occupying, as we argue below, more than *ca.* 99.8% in radius of the convection zone. By definition, the emission and absorption of acoustic waves by turbulence in this shell is negligible, and convection–oscillation coupling can be ignored accurately. Consequently, the anelastic condition can be applied. Furthermore, *p*-mode amplitudes are much smaller than in the outer shell and the solar gas in this shell is optically thick so radiative damping is negligible. The contribution to the interaction coefficient describing three-mode coupling in the inner shell is relatively small (Kumar & Goldreich 1989). Therefore, we argue that splitting and the global distortion of acoustic wave-fronts dominantly result from convection that is relatively coherent temporally and spatially. If long-lived, long-wavelength features of convection do exist, they would possess characteristic signatures in *p*-mode frequencies and line-widths that could be computed from the theory presented herein. In principle, once we have identified these signatures, their observation would place constraints on the causative convective structures. For example, we show in Lavely & Ritzwoller (1992) that because of the way helioseismic Doppler images are reduced and analysed, the effect of aspherical structure is to broaden line-widths and that within a given multiplet the line-broadening is most pronounced for low (m) states. This effect can be significant for modes with low intrinsic damping rates. The value of helioseismological constraints such as these would be enhanced by the fact that large-scale convection has been linked with the dynamic structure of the differential rotation (Gilman 1987) and with the solar dynamo (Stix 1981). In addition, magnetic activity observed at the solar surface probably is controlled by flows at depth.

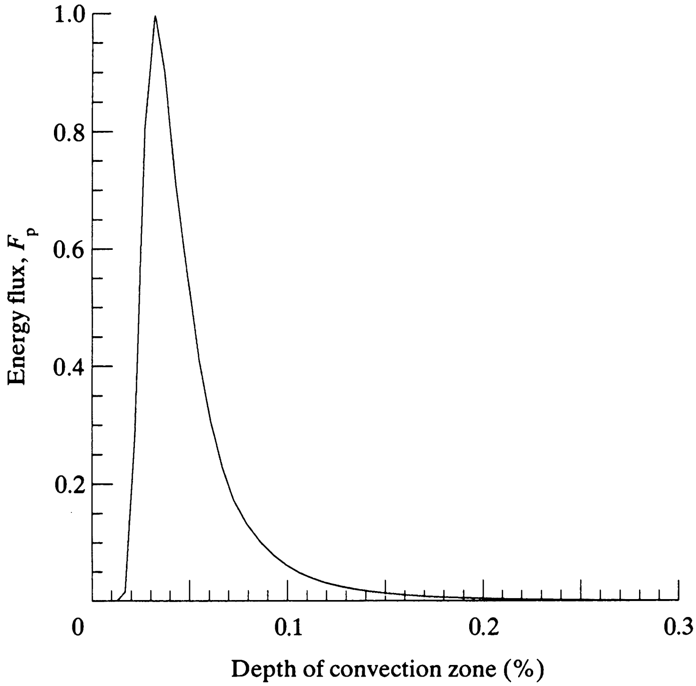

Figure 2. A plot of $F_p = M^{\frac{15}{2}} F_c$ which is the flux of energy pumped into the acoustic modes from the convective motions (see (12)). We have normalized F_p by its peak value. We use the radial dependence of F_p to argue that coupling between convection and p-modes is significant only in the top *ca.* 0.2% of the convection zone.

Next, we address two questions in §§1*d* and *e*: (1) What is the evidence that large-scale convection exists in the inner shell? (2) What is the extent of the outer shell where we do not accurately model the convective effect of convection?

(*d*) *On the existence of giant-cells*

A problem for the utility of the theory presented here is that giant-cells have not been unambiguously observed; if they do exist, their surface amplitudes are less than 10 m s^{-1} (Howard & LaBonte 1980; LaBonte *et al.* 1981; Brown & Gilman 1984). Nevertheless, the evidence for their existence is strong, though circumstantial. (1) The Sun displays a number of features that are suggestive of sustained large-scale motions (Gilman 1987). These include persistent large-scale patterns in the solar magnetic field, the coronal holes which survive several solar rotation periods without being sheared apart by differential rotation, and the existence of active longitudes where new active regions preferentially arise. (2) The observed distinct cellular convection may continue well below the surface. Under mixing length theory (figure 1), the scale of convective eddies is set by H_p so that one predicts a hierarchy of convective cells with monotonically increasing vertical scale. In addition, both linear and nonlinear models (Gough *et al.* 1976) have shown that even when the fluid is compressible, and the stratification includes several scale heights, convection spanning the entire unstable layer is favoured. Thus, for the Sun, patterns of motion with horizontal dimensions up to the depth of the convection zone (i.e. $\lambda \approx$ 200000 km or harmonic degrees of $l \approx 20$) would be expected. (3) The space-lab

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

experiment of thermal convection (Hart *et al.* 1986) and the numerical simulations of Glatzmaier (1984) and Gilman & Miller (1986) suggest that large and sustained patterns of motion may exist in the Sun with scales approaching the depth of the convection zone. (4) Hill (1988) constructed three-dimensional spectra (k_x, k_y, ω) of helioseismic images of small rectangular regions near the solar equator and discovered relatively large-scale horizontal, poleward flows of approximately $100\ \mathrm{m\ s^{-1}}$ that may be the surface expression of giant-cells. (5) Finally, a possible explanation of the smaller vertical velocities of the supergranules and the absence of a strong signature of giant-cells in the data of Howard & LaBonte (1980) may be found in the work of Latour *et al.* (1981) and van Ballegooijen (1986). Latour *et al.* (1981) found that buoyancy breaking in A-type stars may occur in upward-directed flows that have horizontal scales large compared with the pressure scale height of the region into which they penetrate. This leads to lateral deflection and strong horizontal shearing motions. If this result applies as well to G-type stars such as the Sun, it may provide the explanation for the lack of surface observations of giant-cells. In addition, van Ballegooijen (1986) found that density stratification screens out periodic components of the near surface flow pattern in his convection model so that periodic motions that exist at depth would not be observed at the surface.

(*e*) *Justification of linearization for application to giant-cell convection*

We now attempt to quantify the extent of the outer shell, defined to be that region where convection–oscillation coupling is appreciable. The extent of energy exchange between oscillations and turbulent convection depends on their relative time and velocity scales. Perhaps the best available measure of the coupling between convection and acoustic oscillations is the flux of energy F_p pumped into the acoustic modes from the convective motions. Goldreich & Kumar (1990) derive an expression for F_p given by

$$F_p = M^{15/2} F_c, \tag{12}$$

where M and F_c are defined, respectively, in equations (8) and (11).

An inspection of figure 2, which plots the radial dependence of F_p, reveals that convection–oscillation coupling is relatively insignificant below the top *ca.* 0.15% of the convection zone. Thus, as a mechanism of oscillation–convection coupling, Reynolds stresses and entropy fluctuations act far more efficiently in the top few scale heights than in the deeper layers where the characteristic velocities are smaller and the length scales are larger.

Kumar & Goldreich (1989) also discuss the effect of nonlinear interactions among solar acoustic modes. They argue that these interactions are strongest in the outermost layers of the Sun. Indeed, an inspection of their figure 3 indicates that the coupling coefficients are sensitive to three-mode interactions only in the outer *ca.* 0.2% by radius of the convection zone. Consequently, we infer that three-mode interactions can be ignored in the determination of seismic effects of convective flows below this depth.

In criticism of the linearization in both convective velocity and acoustic displacement, it might be suggested on intuitive grounds that a theory governing the effect of convection on acoustic waves must be accurate along the entire path of the acoustic wave, and since all acoustic waves propagate through the outer shell the theory must be general enough to govern outer shell physics. This would certainly be true if we were interested in describing all of the seismic effects of convection.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

However, as discussed in §1*c*, turbulent convection and other nonlinear processes in the outer shell will dominantly affect modal amplitudes and degenerate frequencies. We are only interested here in determining the eigenfunctions and split frequencies of acoustic modes. Consequently, outer shell physics will be subsequently ignored.

In conclusion, we define the outer shell to have a depth of *ca.* 0.2% of the convection zone and we argue that the seismic effect of convection can be modelled accurately with a linearized theory for flows within the inner *ca.* 99.8% of the convection zone by radius.

(*f*) *Overview*

In §2 we discuss reference frames and the separability of the equations of motion, and present a means of transferring the theoretical results presented in this paper from a frame corotating with the Sun to an inertial frame that can be roughly identified as the observer's frame. In §3 we present mathematical representations for convection and for the asphericities in the elastic-gravitational variables. The equations of motion governing the acoustic oscillations in the presence of a steady-state global-scale velocity field and the associated static structural perturbations to density and bulk modulus are derived in §4. We derive in §5 the quasi-degenerate perturbation theory needed to calculate the influence of a velocity field and structural perturbations on solar oscillations. In §6, we derive the general matrix elements that determine the displacement field and split frequencies caused by an anelastic model of convection represented with scalar and vector spherical harmonics. In §7 we discuss properties of the supermatrix. In §8, we consider differential rotation. In §9 we show how theoretical wavefields can be computed for SNRNMAIS and non-SNRNMAIS solar models. The principal conclusions of the paper are summarized in §10.

The system of equations that governs the modal eigenfunctions and eigenfrequencies of the SNRNMAIS solar model is presented in Appendix A. The equation of motion of the non-SNRNMAIS solar model is derived in Appendix B. In Appendix C, we present a mathematical method adapted from Phinney & Burridge (1973) that considerably simplifies the application of differential operators to vector and tensor fields in a spherical geometry which are common in helioseismology. This technique is used to calculate the general matrix element presented in §6. Appendix D outlines the incorporation of the anelastic condition into the general matrix. Appendix E outlines the derivation of matrix elements for aspherical perturbations in the elastic-gravitational variables.

2. Reference frames and the separation of the equations of motion

Helioseismic oscillations are currently observed from the Earth's surface. Space-based measurements will soon exist, but whether measurements are obtained from the Earth or from Space, observers require theoretical results reported in a frame other than the Sun's. In this paper we refer to three reference frames: a frame we call the corotating frame that rotates with the average observed angular rotation rate Ω of the solar surface, a frame we call the observer's frame where helioseismic images are observed, and the inertial frame. The observer's frame is not an inertial frame because of the Earth's rotation, its orbital motion, and because of the acceleration of the solar system. However, since helioseismic data are processed routinely to

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

remove the first two of these effects and since the final effect is small, we will subsequently identify the observer's frame with an inertial frame which is considered to be at rest.

(*a*) *On the separability of the equations of motion*

If solar convection were perfectly axisymmetric, as is differential rotation, the seismic equations of motion could be solved either in the corotating or in the inertial frame. Since axisymmetric flows are stationary relative to both frames, solutions to these equations separate in both frames, that is:

$$\boldsymbol{s}(\boldsymbol{r},t) = \boldsymbol{\Psi}(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\omega t}, \tag{13}$$

where $\boldsymbol{s}(\boldsymbol{r},t)$ is a mode of the model, the eigenfunction is $\boldsymbol{\Psi}(\boldsymbol{r})$ and the angular frequency of the mode is ω. However, in the presence of non-axisymmetric structures or flows, the equations of motion will not separate in both frames. For example, consider the seismic effect of a single convective feature, say a horizontally polarized convective vortex corotating with the average rotation rate of the Sun. In the inertial frame, the convective state of the Sun appears to change with time. Consequently, a seismic modal displacement pattern represented with time invariant basis functions in the inertial frame would itself vary in time. Thus solutions to the equations of motion will not separate in the inertial frame. However, this convective vortex is stationary relative to an observer in the corotating frame and the equations do separate in this frame. For this reason, we present and solve the seismic equations of motion in the corotating frame.

The Sun is not as simple as this convective model with a single vortex locked into the corotating frame. Convective features evolve in time and also interact. In particular, differential rotation would act kinematically to shear convective features. For example, sectoral giant-cells or banana-cells would be sheared by differential rotation as, say, an array of vortices initially aligned latitudinally would become misaligned. Consequently, general convective features are not stationary even relative to the corotating frame so that the equations of motion will not separate in this frame either.

Currently, our analysis requires the use of equation (13). Thus, as discussed in §1*b*, the theory presented here governs only flows and structures that are steady-state relative to the corotating frame. In the sequel all equations will be derived and solved in the corotating frame. However, we desire to present modal eigenfrequencies and eigenfunctions in the inertial frame to facilitate comparison with helioseismic data. We present in §2*b* the necessary transformation between the corotating and inertial frames and discuss briefly how modal frequency measurements made in the inertial frame differ from those made in the corotating frame.

(*b*) *Transforming from a corotating to an inertial frame*

The major result of this paper is analytical expressions for the eigenfrequencies and eigenfunctions of a non-SNRNMAIS solar model. We wish to find a way to transform these expressions, derived in the corotating frame, into the inertial frame. Let us define two sets of spherical polar coordinates relative to which the following position vectors are defined: $\boldsymbol{r}_{\mathrm{R}} = (r_{\mathrm{R}}, \theta_{\mathrm{R}}, \phi_{\mathrm{R}})$ in the corotating frame and $\boldsymbol{r}_{\mathrm{I}} = (r_{\mathrm{I}}, \theta_{\mathrm{I}}, \phi_{\mathrm{I}})$ in the inertial frame, where θ and ϕ are colatitude and longitude, respectively. We wish to derive a means of transforming the mode $\tilde{\boldsymbol{s}}_j^{\mathrm{R}}(\boldsymbol{r}_{\mathrm{R}},t)$ (with index j) of the non-SNRNMAIS solar model in the corotating frame into $\tilde{\boldsymbol{s}}_j^{\mathrm{I}}(\boldsymbol{r}_{\mathrm{I}},t)$ the mode in the inertial frame.

First, note that the colatitudinal and azimuthal angles θ_{I} and ϕ_{I} are related to θ_{R} and ϕ_{R} in the following way:

$$\theta_{\mathrm{I}} = \theta_{\mathrm{R}}, \tag{14}$$

$$\phi_{\mathrm{I}} = \phi_{\mathrm{R}} + \Omega t. \tag{15}$$

the functional dependences of $\tilde{\boldsymbol{s}}_j^{\mathrm{R}}$ and $\tilde{\boldsymbol{s}}_j^{\mathrm{I}}$ on colatitude and longitude are identical so that by equations (14) and (15):

$$\tilde{\boldsymbol{s}}_j^{\mathrm{I}}(\boldsymbol{r}_{\mathrm{I}}, t) = \tilde{\boldsymbol{s}}_j^{\mathrm{I}}(r, \theta_{\mathrm{I}}, \phi_{\mathrm{I}}, t) = \tilde{\boldsymbol{s}}_j^{\mathrm{R}}(r, \theta_{\mathrm{R}}, \phi_{\mathrm{R}}, t) = \tilde{\boldsymbol{s}}_j^{\mathrm{R}}(r, \theta_{\mathrm{I}}, \phi_{\mathrm{I}} - \Omega t, t). \tag{16}$$

As will be shown in §5, the mode $\tilde{\boldsymbol{s}}_j^{\mathrm{R}}(\boldsymbol{r}_{\mathrm{R}}, t)$ in the corotating frame can be written as a linear combination of the eigenfunctions $\boldsymbol{s}_k$ of the SNRNMAIS modes defined in equation (1):

$$\begin{aligned}\tilde{\boldsymbol{s}}_j^{\mathrm{R}}(\boldsymbol{r}_{\mathrm{R}}, t) &= \left[\sum_{k \in K} a_k^j\, \boldsymbol{s}_k(\boldsymbol{r}_{\mathrm{R}})\right] \mathrm{e}^{\mathrm{i}\omega_j t} \\ &= \boldsymbol{\Psi}_j^{\mathrm{R}}(\boldsymbol{r}_{\mathrm{R}})\, \mathrm{e}^{\mathrm{i}\omega_j t}.\end{aligned} \tag{17}$$

The eigenspace K, over which the sum is taken, is defined to contain only those SNRNMAIS eigenfunctions required to represent the eigenfunction of the perturbed model. The nature of this sum is discussed in detail in §5. In particular, it is shown that the expansion coefficients a_k^j for a mode j are components of an eigenvector of the supermatrix, and the coefficients themselves form a matrix called the eigenvector matrix. The quantity in square brackets is the eigenfunction of mode j with frequency ω_j that we identify as $\boldsymbol{\Psi}_j^{\mathrm{R}}(\boldsymbol{r}_{\mathrm{R}})$ and which is time independent in the corotating frame. Substituting equation (17) into equation (16) yields the desired expression for the mode in the inertial frame:

$$\begin{aligned}\tilde{\boldsymbol{s}}_j^{\mathrm{I}}(\boldsymbol{r}_{\mathrm{I}}, t) = \tilde{\boldsymbol{s}}_j^{\mathrm{R}}(r, \theta_{\mathrm{I}}, \phi_{\mathrm{I}} - \Omega t, t) &= \left[\sum_{k \in K} a_k^j\, \boldsymbol{s}_k(r, \theta_{\mathrm{I}}, \phi_{\mathrm{I}} - \Omega t)\right] \mathrm{e}^{\mathrm{i}\omega_j t} \\ &= \left[\sum_{k \in K} a_k^j\, \boldsymbol{s}_k(r, \theta_{\mathrm{I}}, \phi_{\mathrm{I}})\, \mathrm{e}^{-\mathrm{i}m\Omega t}\right] \mathrm{e}^{\mathrm{i}\omega_j t} \\ &= \boldsymbol{\Psi}_j^{\mathrm{I}}(\boldsymbol{r}_{\mathrm{I}}, t)\, \mathrm{e}^{\mathrm{i}\omega_j t},\end{aligned} \tag{18}$$

where we have used equations (14) and (15), the definition of a SNRNMAIS eigenfunction given by equation (1), and the fact that the azimuthal dependence of a spherical harmonic $Y_l^m(\theta, \phi)$ is given by $\exp(\mathrm{i}m\phi)$.

A comparison of equations (17) and (18) reveals an important difference between the eigenfunctions in the corotating and inertial frames. In the inertial frame, the spatial eigenfunctions $\boldsymbol{\Psi}_j^{\mathrm{I}}(\boldsymbol{r}_{\mathrm{I}}, t)$ are themselves time dependent and provide a slowly varying envelope function in addition to the harmonic oscillation with frequency ω_k, whereas in the corotating frame the spatial eigenfunctions $\boldsymbol{\Psi}_j^{\mathrm{R}}(\boldsymbol{r}_{\mathrm{R}})$ are time invariant. Thus, a mode of oscillation, which by definition has a single frequency in the corotating frame, becomes multiply periodic in the inertial frame.

For clarity, we consider as an example the case of self-coupling where the eigenspace K is spanned by the $(2l+1)$ SNRNMAIS eigenfunctions of the multiplet ${}_nS_l$. In this case, the eigenfunction in equation (18) can be rewritten

$$\boldsymbol{\Psi}_j^{\mathrm{I}}(\boldsymbol{r}_{\mathrm{I}}, t) = \sum_{m=-l}^{l} a_{(n,l,m)}^j\, \boldsymbol{s}_{(n,l,m)}(\boldsymbol{r}_{\mathrm{I}})\, \mathrm{e}^{-\mathrm{i}m\Omega t}, \tag{19}$$

so that in the inertial frame there are $(2l+1)$ frequencies, separated by Ω, associated with every eigenfunction. The extra 'modes' are a reference frame effect, and are not an intrinsic property of the Sun. However, if all convective flows were perfectly axisymmetric, like differential rotation, then the expansion coefficients $a^j_{(n,l,m)}$ would become Kronecker delta functions δ_{jm} (alternately, the eigenvector matrix would be the identity matrix) and the eigenfunction in the inertial frame could be represented with a SNRNMAIS eigenfunction with a simple time dependence:

$$\boldsymbol{\Psi}^{\mathrm{I}}_m(\boldsymbol{r}_{\mathrm{R}}, t) = \boldsymbol{s}_{(n,l,m)}(\boldsymbol{r}_{\mathrm{I}})\,\mathrm{e}^{-\mathrm{i}m\Omega t}. \tag{20}$$

Perhaps a more relevant way of viewing all of this in the inertial frame would be not in terms of the spatial pattern that oscillates with a single frequency, though this is the way we have defined a mode of oscillation, but rather in terms of the time dependence or spectrum of a single spatial pattern. For clarity, let us continue to work within the self-coupling approximation. Consider a spatial pattern given by a single spherical harmonic $Y^{m'}_l(\theta, \phi)$, the basis function onto which observers frequently project helioseismic data. Then, we wish to determine the time dependence of the basis function $\boldsymbol{s}_{(n,l,m')}(\boldsymbol{r}_{\mathrm{I}})$. This is determined by summing the modes in (18) over the $(2l+1)$ components of the multiplet and retaining only contributions to $\boldsymbol{s}_{(n,l,m')}(\boldsymbol{r}_{\mathrm{I}})$:

$$\begin{aligned}\sum_{j=1}^{2l+1} \tilde{\boldsymbol{s}}^{\mathrm{I}}_j(\boldsymbol{r}_{\mathrm{I}}, t)\,\delta_{m'm} &= \sum_{j=1}^{2l+1} \boldsymbol{\Psi}^{\mathrm{I}}_j(\boldsymbol{r}_{\mathrm{I}})\exp{(\mathrm{i}\omega_j t)}\,\delta_{m'm} \\ &= \left[\sum_{j=1}^{2l+1} a^j_{m'}\exp{(\mathrm{i}\delta\omega_j t)}\right]\boldsymbol{s}_{(n,l,m')}(\boldsymbol{r}_{\mathrm{I}})\exp{(\mathrm{i}(\omega_{nl}-m'\Omega)\,t)} \\ &= \boldsymbol{\Phi}_{m'}(t)\,\boldsymbol{s}_{(n,l,m')}(\boldsymbol{r}_{\mathrm{I}})\exp{(\mathrm{i}(\omega_{nl}-m'\Omega)\,t)},\end{aligned} \tag{21}$$

where we have used equation (19) in the penultimate summation, and set the eigenfrequency $\omega_j = w_{nl}+\delta\omega_j$, where ω_{nl} is the degenerate frequency of the multiplet and $\delta\omega_j$ is the intrinsic frequency perturbation of mode j caused by structural asphericities and convective flows.

Projection onto a single spherical harmonic component would yield a spectrum that is the Fourier transform of $\boldsymbol{\Phi}_{m'}(t)\exp{(\mathrm{i}(\omega_{nl}-m'\Omega)\,t)}$, a spectrum composed of a cluster of $(2l+1)$ closely spaced peaks centred at frequency $(\omega_{nl}-m'\Omega)$. If the self-coupling approximation were not invoked, then the functional form of equation (21) would be the same but the cluster of peaks would contain more than $(2l+1)$ elements.

3. Parametrization of convective flow and asphericities in the elastic-gravitational variables

All convective and structural perturbations are defined relative to the SNRNMAIS model. The perturbations include rigid rotation, a convective velocity field that is stationary in the corotating frame, and static perturbations to the elastic-gravitational variables. In this section we present the parametrization of each of these perturbations.

(a) *The velocity field and the rotation vector*

The velocity field $\boldsymbol{u}_0$ is defined with respect to the corotating frame and is defined to be stationary in this frame as well. Thus $\boldsymbol{u}_0$ does not contain a contribution due to the velocity field $(\boldsymbol{v} = \boldsymbol{\Omega}\times\boldsymbol{r})$ of rigid rotation. Rigid rotation is included when the results are transformed back to the inertial frame as discussed in §2*b*. We later require an expression for $\boldsymbol{\Omega}$:

$$\boldsymbol{\Omega} = \Omega \cos\theta \hat{\boldsymbol{r}} - \Omega \sin\theta \hat{\boldsymbol{\theta}}, \tag{22}$$

where Ω is the average angular rotation rate at the solar surface. The velocity field $\boldsymbol{u}_0$ can be decomposed into a sum of poloidal $\boldsymbol{P}$ and toroidal $\boldsymbol{T}$ vector spherical harmonics:

$$\boldsymbol{u}_0(\boldsymbol{r}) = \sum_{s=0}^{\infty} \sum_{t=-s}^{s} \boldsymbol{P}_s^t(r, \theta, \phi) + \boldsymbol{T}_s^t(r, \theta, \phi). \tag{23}$$

The poloidal and toroidal components are fully characterized by the radius dependent vector spherical harmonic expansion coefficients $u_s^t(r)$, $v_s^t(r)$, and $w_s^t(r)$:

$$\boldsymbol{P}_s^t(r, \theta, \phi) = u_s^t(r)\, Y_s^t(\theta, \phi)\, \hat{\boldsymbol{r}} + v_s^t(r)\, \nabla_1\, Y_s^t(\theta, \phi), \tag{24}$$

$$\boldsymbol{T}_s^t(r, \theta, \phi) = -w_s^t(r)\, \hat{\boldsymbol{r}} \times \nabla_1\, Y_s^t(\theta, \phi), \tag{25}$$

where ∇_1 is the surface gradient operator given by equation (2) and the normalization of the spherical harmonics is given by equation (3).

Consistent with formal assumption (3) in §1*b*, the flow field must satisfy the anelastic condition:

$$\nabla \cdot (\rho_0 \boldsymbol{u}_0) = 0, \tag{26}$$

which implies that

$$\partial_r (r^2 \rho_0 u_s^t) = \rho_0\, r s(s+1)\, v_s^t \tag{27}$$

for each s and t in $\boldsymbol{u}_0$. The reality of the flow fields implies that the coefficients in equation (23) satisfy the conditions $u_s^{-t} = (-1)^t u_s^{t*}, v_s^{-t} = (-1)^t v_s^{t*}$, and $w_s^{-t} = (-1)^t w_s^{t*}$. In addition, to eliminate convective overshoot at the surface, and to insure that the general matrix $\boldsymbol{C}_{n'n,l'l}$ to be derived in §6 is hermitian (see equation (32) of Lynden-Bell & Ostriker 1967), the radial component of the velocity field must vanish at the surface. Therefore, we require

$$u_s^t(R_\odot) = 0 \tag{28}$$

for each s and t.

(*b*) *Perturbations to the elastic-gravitational variables*

The structural perturbations $\delta\kappa_0$, $\delta\rho_0$, and $\delta\phi_0$ to the elastic-gravitational variables are defined as follows:

$$\kappa(r, \theta, \phi) = \kappa_0(r) + \delta\kappa_0(r, \theta, \phi), \tag{29}$$

$$\rho(r, \theta, \phi) = \rho_0(r) + \delta\rho_0(r, \theta, \phi), \tag{30}$$

$$\phi(r, \theta, \phi) = \phi_0(r) + \delta\phi_0(r, \theta, \phi). \tag{31}$$

In §4*b* we discuss notation intended to differentiate these perturbations from seismically induced perturbations. We consider structural perturbations caused by the rigid rotation of the Sun, namely its hydrostatic ellipticity of figure, separately from other structural perturbations most of which will result from convection. Chandrasekhar & Roberts (1963) have shown the rotation induced perturbations to $\rho_0(r)$ and $\phi_0(r)$ are given by

$$\delta\rho_{\mathrm{e}}(r, \theta) = (\tfrac{4}{5}\pi)^{\frac{1}{2}} \tfrac{2}{3} r \epsilon(r)\, \partial_r \rho_0(r)\, Y_2^0(\theta, \phi), \tag{32}$$

$$\delta\phi_{\mathrm{e}}(r, \theta) = (\tfrac{4}{5}\pi)^{\frac{1}{2}} [\tfrac{2}{3} r \epsilon(r)\, \partial_r \phi_0(r) - \tfrac{1}{3}\Omega^2 r^2]\, Y_2^0(\theta, \phi), \tag{33}$$

where $\epsilon(r)$ is the ellipticity defined in equation (A 11) of Woodhouse & Dahlen (1978). The ellipticity is the solution of Clairaut's equation. (See Tassoul (1978) for a derivation of $\epsilon(r)$.) Assuming that the surfaces of constant κ in the rotating, elliptical Sun coincide with the surfaces of constant ρ, we obtain

$$\delta\kappa_e(r,\theta) = (\tfrac{4}{5}\pi)^{\frac{1}{2}}\tfrac{2}{3}r\epsilon(r)\,\partial_r\kappa_0(r)\,Y_2^0(\theta,\phi). \tag{34}$$

The perturbations $\delta\kappa_e, \delta\rho_e$, and $\delta\phi_e$ are $\mathcal{O}(\Omega^2)$; contributions from the differential rotation are much smaller and thus they have been neglected. Having explicitly accounted for ellipticity of figure, we now expand all remaining quantities in spherical harmonics to obtain

$$\delta\kappa_0(r,\theta,\phi) = (\tfrac{4}{5}\pi)^{\frac{1}{2}}\tfrac{2}{3}r\epsilon(r)\,\partial_r\kappa_0(r)\,Y_2^0(\theta,\phi) + \sum_{s=0}^{\infty}\sum_{t=-s}^{s}\delta\kappa_s^t(r)\,Y_s^t(\theta,\phi), \tag{35}$$

$$\delta\rho_0(r,\theta,\phi) = (\tfrac{4}{5}\pi)^{\frac{1}{2}}\tfrac{2}{3}r\epsilon(r)\,\partial_r\rho_0(r)\,Y_2^0(\theta,\phi) + \sum_{s=0}^{\infty}\sum_{t=-s}^{s}\delta\rho_s^t(r)\,Y_s^t(\theta,\phi), \tag{36}$$

$$\delta\phi_0(r,\theta,\phi) = (\tfrac{4}{5}\pi)^{\frac{1}{2}}[\tfrac{2}{3}r\epsilon(r)\,\partial_r\phi_0(r) - \tfrac{1}{3}\Omega^2 r^2]\,Y_2^0(\theta,\phi) + \sum_{s=0}^{\infty}\sum_{t=-s}^{s}\delta\phi_s^t(r)\,Y_s^t(\theta,\phi), \tag{37}$$

where, of course, the expansion coefficients $\delta\kappa_2^0(r), \delta\rho_2^0(r)$ and $\delta\phi_2^0(r)$ do not include contributions from hydrostatic ellipticity. In §6*c* we will use the fact that, due to Poisson's equation, the coefficients $\delta\phi_s^t(r)$ are not independent of the coefficients $\delta\rho_s^t(r)$ to simplify our final expression for the general matrix element.

4. The equations of motion

In this section we derive the equations of motion that govern the acoustic oscillations of the SNRNMAIS and non-SNRNMAIS solar models. These equations will be presented in the corotating frame. The numerical solution of the two-point boundary value problem described in Appendix A yields the seismic eigenfrequencies and eigenfunctions of the SNRNMAIS solar model. The eigenfrequencies and eigenfunctions of the non-SNRNMAIS solar model will be calculated by applying quasi-degenerate perturbation theory (described in §5). The use of this theory produces the general matrix element (§6). Calculation of the general matrix element requires an equation of motion that accurately governs acoustic oscillations consistent with the formal assumptions listed in §1*b*. The equation of motion appropriate to this purpose is the major product of this section and is given by equation (50) together with equation (B 20) and equations (B 22)–(B 25).

(*a*) *The seismically unperturbed equations of motion*

We present here the seismically unperturbed equations of motion of a rotating, convecting solar model with a general velocity field $\boldsymbol{v}$ and general dependence on the elastic-gravitational variables. The unperturbed mass continuity and conservation of energy equations are given, respectively, by

$$\partial\rho/\partial t + \nabla\cdot(\rho\boldsymbol{v}) = 0, \tag{38}$$

$$\rho T\,\mathrm{D}S/\mathrm{D}t = \text{entropy production terms}, \tag{39}$$

where the total time derivative is defined by

$$\mathrm{D}/\mathrm{D}t = \partial/\partial t + \boldsymbol{v}\cdot\nabla. \tag{40}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The right-hand side of equation (39) represents the production of entropy through dissipative processes such as heat conduction, viscous shearing and expansion, emission and absorption of radiation, divergence of the convective flux, and so on. In the absence of entropy production terms, equation (39) states that entropy must be conserved along streamlines of the motion. In general, such a velocity field cannot be designed *ab initio* and this requirement is not naturally incorporated into the lagrangian describing the oscillations. Rather, we assume in the sequel that the flow field $\boldsymbol{v}$ satisfies *a priori* equation (39); presumably, this will be the case if in the numerical implementation of the theory $\boldsymbol{v}$ is obtained from a self-consistent dynamical calculation, and if in the practical application of the theory, the relevant helioseismic observations are produced by a velocity field that satisfies equation (39). Since in our theory convection and acoustic oscillations do not couple, we can consider the equations governing convection separately from the equations governing the oscillations. In our subsequent discussion of oscillations, we will assume that the convective equations have been solved separately and will assume that the lagrangian variation of the entropy is zero, and, therefore, drop any further consideration of the energy equation. Ignoring magnetic fields, Reynolds stresses, and the effects of external body forces, the conservation of linear momentum in the corotating frame can be written

$$\mathrm{D}\boldsymbol{v}/\mathrm{D}t + 2\boldsymbol{\Omega}\times\boldsymbol{v} = -\rho^{-1}\nabla P - \nabla\Phi, \tag{41}$$

where the solar potential function Φ is defined as

$$\Phi = \phi + \psi_{\mathrm{c}}, \tag{42}$$

and ψ_{c} is the rotational potential due to centripetal acceleration; i.e.

$$\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\boldsymbol{r} = \nabla\psi_{\mathrm{c}}, \tag{43}$$

where $\psi_{\mathrm{c}} = -1/2(\Omega^2 r^2 - (\boldsymbol{\Omega}\cdot\boldsymbol{r})^2)$.

The gravitational potential is governed by Poisson's equation:

$$\nabla^2\phi = 4\pi G\rho. \tag{44}$$

As we consider only adiabatic oscillations in this paper, we take the equation of state to be a function of specific entropy S and density to simplify the calculation of the lagrangian variation of the pressure.

The equilibrium equations of the reference state governed by equations (41) and (44) are given by

$$\rho_0\nabla\Phi_0 = -\nabla P_0, \tag{45}$$

$$\nabla^2\phi_0 = 4\pi G\rho_0, \tag{46}$$

where G is the universal constant of gravitation, $\Phi_0 = \phi_0 + \psi_{\mathrm{c}}$, and we have dropped the self-advection term consistent with formal assumption (3). Equations (45) and (46) are subject to the boundary conditions that ϕ_0, and the radial components of $\nabla\phi_0$ and the traction are continuous across $R_{\odot}$:

$$[\phi_0]_-^+ = 0, \tag{47}$$

$$[\hat{\boldsymbol{r}}\cdot\nabla\phi_0]_-^+ = 0, \tag{48}$$

$$[\hat{\boldsymbol{r}}\cdot\boldsymbol{I}P_0]_-^+ = 0, \tag{49}$$

where $\boldsymbol{I}$ is the identity tensor, and the notation $[\phi_0]_-^+$ denotes the jump discontinuity of ϕ_0, and so forth.

(*b*) *Model notation and lagrangian and eulerian variations due to seismic motion*

Any scalar quantity Q of the SNRNMAIS model is a function of radius alone, and we denote that quantity by Q_0. For example, the elastic-gravitational variables of the SNRNMAIS model are given by $\kappa_0(r)$, $\rho_0(r)$, and $\phi_0(r)$. We specify eulerian perturbations with the eulerian change operator δ. To avoid confusion between eulerian perturbations induced by seismic motion and eulerian elastic-gravitational structural perturbations to the SNRNMAIS model, we introduce the following convention to be followed throughout the paper. The expression δQ signifies the eulerian perturbation of Q due to seismic motion: e.g. $\delta\kappa$, $\delta\rho$, and so on. However, δQ_0 signifies a non-seismic, structural perturbation to Q in the SNRNMAIS model: e.g. $\delta\kappa_0$, $\delta\rho_0$, and so on. In addition, we will use the notation δQ_e to indicate the perturbation to any elastic-gravitational variable Q due to hydrostatic ellipticity of figure. The lagrangian change operator Δ will denote seismic perturbations only.

(*c*) *The equations of motion of the* SNRNMAIS *and non-*SNRNMAIS *solar models*

The equation of motion of the non-SNRNMAIS solar model is derived in Appendix B. Quoting the final result (see (B 26)), we obtain

$$-(\rho_0+\delta\rho_0)\,\omega^2\boldsymbol{s}+\rho_0\,\boldsymbol{T}(\boldsymbol{s}) = \mathscr{L}_0(\boldsymbol{s})+\delta\mathscr{L}_0(\boldsymbol{s}), \tag{50}$$

where $\boldsymbol{T}(\boldsymbol{s})$, $\mathscr{L}_0(\boldsymbol{s})$, and $\delta\mathscr{L}_0(\boldsymbol{s})$ are defined, respectively, by equations (B 22), (B 20), and (B 25). The equation of motion of the SNRNMAIS model may be obtained from equation (50) by setting $\delta\rho_0 = \boldsymbol{T}(\boldsymbol{s}) = \delta\mathscr{L}_0(\boldsymbol{s}) = 0$:

$$-\rho_0\,\omega^2\boldsymbol{s} = \mathscr{L}_0(\boldsymbol{s}). \tag{51}$$

Subject to the appropriate boundary conditions, equation (51) along with the perturbed Poisson equation (B 11) and perturbed continuity equation can be solved to yield the eigenfunctions and eigenfrequencies of the SNRNMAIS solar model; this is the subject of Appendix A.

5. Quasi-degenerate perturbation theory

In general, the determination of the eigenfunctions and eigenfrequencies for a general solar model requires the application of a perturbative, variational, or numerical technique. We have chosen to use quasi-degenerate perturbation theory, as distinct from degenerate perturbation theory, to determine these quantities. The derivation that follows can be applied to a general, non-SNRNMAIS solar model in which the asphericities (e.g. in the elastic-gravitational variables, in convective velocities, and so on) are small and are stationary in the corotating frame. The difference between quasi-degenerate and degenerate perturbation theories lies in the choice of the eigenspace used to represent the eigenfunctions in each theory. In quasi-degenerate perturbation theory, the eigenspace K consists of all eigenfunctions of the SNRNMAIS solar model with nearly identical degenerate frequencies. We call this the quasi-degeneracy condition, which we state quantitatively below in §5*a*. In degenerate perturbation theory, only the eigenfunctions from SNRNMAIS modes that are exactly degenerate compose the eigenspace from which an eigenfunction of the perturbed model is represented. Except in the rare case of an accidental exact degeneracy, within degenerate perturbation theory these eigenfunctions will share the same radial order n and harmonic degree l so that there will be $(2l+1)$ SNRNMAIS eigenfunctions contributing to the linear combination for each eigenfunction of the non-SNRNMAIS solar model. Within quasi-degenerate perturbation theory the basis functions can differ in n and l.

In §§5b and c we derive the principal result of quasi-degenerate perturbation theory, the general matrix element, and show how general matrix elements compose the supermatrix whose eigenvalues and eigenvectors prescribe, respectively, the modal frequencies and eigenfunctions of a non-SNRNMAIS solar model. Our derivation of the general matrix element is general in the sense that we do not specify the nature of the aspherical perturbations, only, as mentioned above, that they are required to be small and stationary in the corotating frame. To the best of our knowledge, Rayleigh's principle has not been proved for quasi-degenerate perturbation theory. Thus we first prove Rayleigh's principle, from which the general matrix element and supermatrix result directly. Finally, in §5d, we specialize the solution to a convecting model with asphericities in the elastic-gravitational variables by making reference to the equation of motion of the non-SNRNMAIS solar model (50). Previous applications of quasi-degenerate perturbation theory in terrestrial seismology can be found in Dahlen (1969), Luh (1974) and Woodhouse (1980).

(a) *The quasi-degeneracy condition*

We motivate the quasi-degeneracy condition by analogy with coupled linear oscillators. A more rigorous motivation based on a property of eigensystems is presented in §7b. It is a general property of all small-amplitude oscillators, one of which we are considering a SNRNMAIS mode to be, that such oscillators only couple strongly if the natural, uncoupled frequencies of the oscillators are nearly degenerate. (Of course, if nonlinear effects are important, oscillators can couple strongly even if their natural, uncoupled frequencies are significantly different.) Thus within our linearized theory, in the presence of a general asphericity the only SNRNMAIS modes that will couple strongly are those that are nearly degenerate. Thus the only eigenfunctions necessary to represent an eigenfunction of the perturbed model are those from modes that are nearly degenerate in the SNRNMAIS solar model. The number of these eigenfunctions and, thus, the dimension of space K, will depend on the desired level of accuracy of the calculation.

We now state the quasi-degeneracy condition quantitively. Define the eigenspace K to comprise the set of all SNRNMAIS eigenfunctions with degenerate frequencies ω_k close to ω_{ref}, our best *a priori* guess of the eigenfrequency of the mode of the non-SNRNMAIS model. Then, an eigenfunction $\boldsymbol{s}_k$ is included in K only if ω_k is such that

$$|\omega_k^2-\omega_{\text{ref}}^2| < \epsilon\tau^2 \quad \text{for} \quad k\in K, \tag{52}$$

where ϵ is a small, fixed, dimensionless parameter, and τ^2 is an angular frequency whose magnitude will determine the level of accuracy of subsequent calculations. In the following, ϵ will only be used as a book-keeping device signifying the accuracy of an equation or the magnitude of an expression and will later be set to unity. For example, the use of ϵ in equation (52) is intended to signify that $|\omega_k^2-\omega_{\text{ref}}^2|$ is $\mathcal{O}(\epsilon)$. Let S denote the set of all SNRNMAIS eigenfunctions and let $K^{\perp}$ denote the complement of K in S. Clearly, for small $\epsilon\tau^2$, dim $(K^{\perp}) \gg$ dim (K), where dim denotes the dimension of the eigenspace.

There are two primary considerations that govern the size of K. (1) The first is the value of τ^2, the choice of which will depend on the desired accuracy of the computation. In practice, τ^2 is chosen such that contributions to the perturbed eigenfunctions from modes outside of the eigenspace determined by the quasi-degeneracy condition can be ignored. (2) The second consideration is that each eigenfunction admitted to the eigenspace K should satisfy the appropriate angular selection rules which are listed in §7*c*.

(*b*) *Statement and proof of Rayleigh's principle for quasi-degenerate perturbation theory*

The use of quasi-degenerate perturbation theory to derive the general matrix element requires Rayleigh's principle. The precise statement of Rayleigh's principle requires the introduction of the following notation. We generalize the equation of motion for a mode k of the SNRNMAIS solar model with eigenfrequency ω_{nl},

$$-\rho_0 \omega_{nl}^2 \boldsymbol{s}_k = \mathscr{L}_0(\boldsymbol{s}_k), \tag{53}$$

by introducing the following perturbation expansions into equation (53):

$$\mathscr{L}_0 \rightarrow \mathscr{L}_0 + \epsilon \mathscr{L}_1, \tag{54}$$

$$\omega_{nl}^2 \rightarrow \omega_{\text{ref}}^2 + \epsilon \omega_1^2, \tag{55}$$

$$\rho_0 \rightarrow \rho_0 + \epsilon \rho_1, \tag{56}$$

$$\boldsymbol{s}_k \rightarrow \tilde{\boldsymbol{s}}_0 + \epsilon \tilde{\boldsymbol{s}}_1. \tag{57}$$

A mode of the non-SNRNMAIS solar model will have a single frequency in the corotating frame. Since the eigenfunctions spanning K have frequencies close to ω_{ref}, we express the squared frequency of the mode of the perturbed model with equation (55) where it should be understood that the eigenfrequency perturbation ω_1^2 can be either positive or negative. The operator $\mathscr{L}_0$ in equation (53) governs the seismic oscillations of the SNRNMAIS model and should be identified with the operator $\mathscr{L}_0$ in equation (B 20). We let $\epsilon \mathscr{L}_1$ denote an arbitrary operator that accounts for departures from a SNRNMAIS solar model. In §5*c*, we specify its form for convective fields with asphericities in the elastic-gravitational variables by using equation (50). Substituting the perturbation expansions in equations (54)–57) into equation (53), we obtain

$$-(\rho_0 + \epsilon \rho_1)(\omega_{\text{ref}}^2 + \epsilon \omega_1^2)(\tilde{\boldsymbol{s}}_0 + \epsilon \tilde{\boldsymbol{s}}_1) = (\mathscr{L}_0 + \epsilon \mathscr{L}_1)(\tilde{\boldsymbol{s}}_0 + \epsilon \tilde{\boldsymbol{s}}_1). \tag{58}$$

Rayleigh's principle governs which terms in equation (58) should be retained if we desire to obtain eigenfrequencies accurate to $\mathcal{O}(\epsilon)$. It states that $\mathcal{O}(\epsilon)$ eigenfunction perturbations $\epsilon \tilde{\boldsymbol{s}}_1$, produce $\mathcal{O}(\epsilon^2)$ eigenfrequency perturbations. That is, Rayleigh's principle states that the first-order eigenfrequencies are stationary to first-order variations in the eigenfunctions. Thus to calculate perturbed eigenfrequencies to $\mathcal{O}(\epsilon)$ requires the use of the first-order perturbation equations (54)–(56), but the displacement eigenfunction in equation (57) need only be retained to $\mathcal{O}(\epsilon^0)$; i.e. equation (57) can be rewritten $\boldsymbol{s}_k \rightarrow \tilde{\boldsymbol{s}}_0$.

We now prove Rayleigh's principle for quasi-degenerate perturbation theory by showing that terms $\epsilon \tilde{\boldsymbol{s}}_1$ can be neglected in calculating the perturbed eigenfrequency to $\mathcal{O}(\epsilon)$. First, retaining terms in equation (58) to $\mathcal{O}(\epsilon)$, we obtain

$$\boldsymbol{0} = (\mathscr{L}_0 + \rho_0 \omega_{\text{ref}}^2)\epsilon \tilde{\boldsymbol{s}}_1 + (\epsilon \mathscr{L}_1 + \epsilon \rho_1 \omega_{\text{ref}}^2)\tilde{\boldsymbol{s}}_0 + \rho_0 \epsilon \omega_1^2 \tilde{\boldsymbol{s}}_0 + (\mathscr{L}_0 + \rho_0 \omega_{\text{ref}}^2)\tilde{\boldsymbol{s}}_0 + \mathcal{O}(\epsilon^2). \tag{59}$$

Within degenerate perturbation theory, the term $(\mathscr{L}_0 + \rho_0 \omega_{\text{ref}}^2)\tilde{\boldsymbol{s}}_0$ is transformed to $(\mathscr{L}_0 + \rho_0 \omega_{nl}^2)\boldsymbol{s}_k = 0$, but within quasi-degenerate perturbation it is of $\mathcal{O}(\epsilon)$, so it is retained. Second, we expand the zero- and first-order eigenfunctions in terms of the SNRNMAIS eigenfunctions. The zero-order eigenfunction $\tilde{\boldsymbol{s}}_0$ is, by the quasi-degeneracy condition, a linear combination of the SNRNMAIS eigenfunctions $\boldsymbol{s}_k$ for $k \in K$, and the first-order eigenfunction perturbations are represented by eigenfunctions within $K^{\perp}$. Thus we write

$$\tilde{\boldsymbol{s}}_0 = \sum_{k \in K} a_k \boldsymbol{s}_k, \tag{60}$$

$$\epsilon \tilde{\boldsymbol{s}}_1 = \sum_{k \in K^{\perp}} b_k \boldsymbol{s}_k, \tag{61}$$

where a_k and b_k are complex constants.

Now, following the Galerkin method, we substitute equations (60) and (61) into equation (59), form the inner product of the resulting expression with $\boldsymbol{s}_{k'}^*$, and integrate over the volume of the solar model to obtain

$$0 = \sum_{k \in K^{\perp}} \epsilon b_k \int \boldsymbol{s}_{k'}^* \cdot (\mathscr{L}_0 + \rho_0 \omega_{\text{ref}}^2) \boldsymbol{s}_k \, \mathrm{d}^3\boldsymbol{r} + \sum_{k \in K} a_k \left\{ \int \boldsymbol{s}_{k'}^* \cdot \epsilon (\mathscr{L}_1 + \rho_1 \omega_{\text{ref}}^2) \boldsymbol{s}_k \, \mathrm{d}^3\boldsymbol{r} + \epsilon \omega_1^2 \int \rho_0 \boldsymbol{s}_{k'}^* \cdot \boldsymbol{s}_k \, \mathrm{d}^3\boldsymbol{r} + \int \boldsymbol{s}_{k'}^* \cdot (\mathscr{L}_0 + \rho_0 \omega_{\text{ref}}^2) \boldsymbol{s}_k \, \mathrm{d}^3\boldsymbol{r} \right\}. \tag{62}$$

Setting ϵ to unity, and using equation (5), we find that equation (62) can be rewritten

$$-\sum_{k \in K^{\perp}} b_k (\omega_{\text{ref}}^2 - \omega_{nl}^2) \delta_{k'k} + \sum_{k \in K} a_k Z_{k'k} = \sum_{k \in K} a_k \omega_1^2 \delta_{k'k}, \tag{63}$$

where

$$\left. \begin{aligned} Z_{k'k} &= \frac{1}{N_k} \left\{ \int \boldsymbol{s}_{k'}^* \cdot (-\mathscr{L}_1 - \rho_1 \omega_{\text{ref}}^2) \boldsymbol{s}_k \, \mathrm{d}^3\boldsymbol{r} - (\omega_{\text{ref}}^2 - \omega_{nl}^2) N_K \delta_{k'k} \right\} \quad \text{for} \quad (k', k) \in K. \\ Z_{k'k} &= 0 \quad \text{for} \quad (k', k) \notin K. \end{aligned} \right\} \tag{64}$$

The matrix $\boldsymbol{Z}$ (with components given by $Z_{k'k}$) is called the supermatrix. Were it not for the first term in equation (63), equations (63) and (64) would represent an eigenvalue problem for the eigenfrequency perturbations ω_1^2 and the zero-order eigenfunction expansion coefficients a_k. This term cannot be neglected on the basis of its magnitude since for $k \in K^{\perp}$, $(\omega_{\text{ref}}^2 - \omega_{nl}^2)$ is not guaranteed to be of $\mathscr{O}(\epsilon)$. In fact, in general it can be arbitrarily large. However, terms in b_k for $k \in K^{\perp}$ are non-zero only for $k' = k$ and they fall outside of the matrix of terms in a_k for $(k, k' \in K)$. Therefore, the terms in b_k for $k \in K^{\perp}$ decouple from the terms in a_k for $k \in K$, and it is possible to perform an eigenvalue–eigenvector decomposition of the supermatrix $\boldsymbol{Z}$. Thus, the terms in b_k do not effect the values of the perturbed eigenfrequencies ω_1^2 nor the values of the a_k coefficients. Equation (63) can be rewritten as the eigenvalue–eigenvector problem for ω_1^2 and a_k:

$$\sum_{k \in K} a_k Z_{k'k} = \sum_{k \in K} a_k \omega_1^2 \delta_{n'n} \delta_{l'l} \delta_{m'm}. \tag{65}$$

This completes the proof of Rayleigh's principle; perturbed eigenfrequencies can be computed to $\mathscr{O}(\epsilon)$ by neglecting terms in b_k. This results from the definition of the quasi-degeneracy condition. In practice, the eigenspace K is chosen such that the frequency difference term $(\omega_{\text{ref}}^2 - \omega_{nl}^2)$ in equation (63) is sufficiently large so that b_k is small enough to be ignored.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

(c) *The general matrix element and the supermatrix within quasi-degenerate perturbation theory*

During the proof of Rayleigh's principle, we obtained the general matrix element and the supermatrix. In fact, equation (65) is the principal result of quasi-degenerate perturbation theory. The eigenvalue components of the supermatrix $\boldsymbol{Z}$ are given by the a_k coefficients and the eigenvalues are the eigenfrequency perturbations ω_1^2. Inspection of equation (64) suggests that the general matrix element should be defined by

$$H^{m'm}_{n'n,\,l'l} = \int \boldsymbol{s}^*_{k'} \cdot (-\mathscr{L}_1(\boldsymbol{s}_k) - \rho_1\,\omega^2_{\text{ref}}\,\boldsymbol{s}_k)\,\mathrm{d}^3\boldsymbol{r}, \tag{66}$$

where $H^{m'm}_{n'n,\,l'l}$ is the (m', m) component of the general matrix $\boldsymbol{H}_{n'n,\,l'l}$ for $-l' \leqslant m' \leqslant l'$ and $-l \leqslant m \leqslant l$. Thus the component $Z_{k'k}$ of the supermatrix $\boldsymbol{Z}$ is simply

$$Z_{k'k} = (1/N_k)\{H^{m'm}_{n'n,\,l'l} - (\omega^2_{\text{ref}} - \omega^2_{nl})\,N_k\,\delta_{k'k}\} \tag{67}$$

for $(k', k) \in K$.

With equation (67), equation (65) can be written as the eigensystem:

$$\boldsymbol{Z}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{\Lambda}, \tag{68}$$

where $\boldsymbol{\Lambda}$ is the diagonal matrix of squared eigenfrequency perturbations, $\Lambda_{kj}\,\delta_{kj} = (\omega_1^2)_j$, and $\boldsymbol{A}$ is the eigenvector matrix, each column of which is composed of the expansion coefficients for an eigenfunction: $A_{kj} = a^j_k$. Since the anelastic condition has been applied to the flow field, $\boldsymbol{Z}$ is hermitian:

$$\boldsymbol{Z} = \boldsymbol{A}\boldsymbol{\Lambda}\boldsymbol{A}^{\dagger}, \tag{69}$$

where the superscript † denotes hermitian transpose. The squared frequencies of the perturbed system are given by

$$\omega_j^2 = \omega^2_{\text{ref}} + (\omega_1^2)_j \tag{70}$$

and $1 \leqslant j \leqslant \dim(K)$. The approximation

$$(\omega_1^2)_j \approx 2\omega_{\text{ref}}\,\delta\omega_j \tag{71}$$

may be used so that

$$\omega_j \approx \omega_{\text{ref}} + \delta\omega_j. \tag{72}$$

Thus, the zero-order eigenfunction of the perturbed system is given by equation (60) and the expansion coefficients a^j_k are simply the components of the eigenvectors associated with each eigenfrequency ω_j. Factors of frequency such as ω or ω_{nl} appearing in the general matrix elements that arise, for example, from the Coriolis and advective terms in the equations of motion, should be replaced with ω_{ref}.

Degenerate perturbation theory can be recovered from equation (65) by choosing $\omega_{\text{ref}} = \omega_{nl}$, setting $n' = n$, $l' = l$, and by defining K to include only those SNRNMAIS eigenfunctions that have radial order n and harmonic degree l (i.e. $\dim(K) = 2l+1$).

(d) *The general matrix element for the non-*SNRNMAIS *solar model*

We presently specialize quasi-degenerate perturbation theory to the non-SNRNMAIS solar model (see (50)). The equation of motion governing the oscillations of this model may be rewritten as

$$-(\rho_0 + \delta\rho_0)\,(\omega^2_{\text{ref}} + \omega_1^2)\,(\tilde{\boldsymbol{s}}_0 + \tilde{\boldsymbol{s}}_1) = (\mathscr{L}_0 + \delta\mathscr{L}_0 - \rho_0\,\boldsymbol{T})\,(\tilde{\boldsymbol{s}}_0 + \tilde{\boldsymbol{s}}_1). \tag{73}$$

Comparing the above equation to equation (58), we may make the identifications

$$\mathscr{L}_1 = \delta\mathscr{L}_0 - \rho_0\,\boldsymbol{T}, \tag{74}$$

$$\rho_1 = \delta\rho_0. \tag{75}$$

The general matrix element for the non-SNRNMAIS solar model can be obtained by substituting equations (74) and (75) into equation (66) to obtain

$$H^{m'm}_{n'n,l'l} = \int \boldsymbol{s}^*_{k'}\cdot(\rho_0\,\boldsymbol{T}(\boldsymbol{s}_k) - \delta\mathscr{L}_0(\boldsymbol{s}_k) - \delta\rho_0\,\omega^2_{\mathrm{ref}}\,\boldsymbol{s}_k)\,\mathrm{d}^3\boldsymbol{r}. \tag{76}$$

with the (k', k) component of the supermatrix $\boldsymbol{Z}$ given by equation (67). The explicit form of $H^{m'm}_{n'n,l'l}$ written in terms of the scalar eigenfunctions of the SNRNMAIS solar model is presented in §6 and derived in Appendixes D and E. The construction of the supermatrix $\boldsymbol{Z}$ in terms of the general matrix elements is discussed further in §7.

6. The general matrix element

The general matrix element $H^{m'm}_{n'n,l'l}$ (equation (76)) determines the strength of coupling between the SNRNMAIS modes $k = (n, l, m)$ and $k' = (n', l', m')$. Following the discussion in §5, the supermatrix $\boldsymbol{Z}$ (equation (67)) is composed of an assemblage of general matrix elements. The eigenvalues of $\boldsymbol{Z}$ are simply related to the eigenfrequencies of the convecting model with aspherical perturbations in the elastic-gravitational variables. The eigenvector components of $\boldsymbol{Z}$ are simply the expansion coefficients of the eigenfunctions of the perturbed model; the SNRNMAIS modal eigenfunctions act as basis functions. In this section we derive explicit expressions in terms of the scalar eigenfunctions of the SNRNMAIS model for the general matrix elements of a non-SNRNMAIS solar model with convective flow and aspherical perturbations in the elastic-gravitational variables.

Our approach to the derivation of the general matrix element differs from the approach used by Woodhouse & Dahlen (1978) and Woodhouse (1980). Woodhouse & Dahlen (1978) applied Rayleigh's principle to derive the matrix elements for an aspherical Earth model. Due to the presence of internal discontinuities in Earth models (e.g. the core-mantle boundary), aspherical perturbations to these boundaries have been modelled. By using Rayleigh's principle, the origin of the extra terms required to account for the additional degrees of freedom is clearly evident (Woodhouse 1976; Dahlen 1976). A proper accounting of deformation to internal discontinuities introduces considerable complexity to the calculation. Since there are no first-order discontinuities in the standard solar model (though there is a discontinuity in the derivative of the sound speed at the base of the convection zone), we have not used the general technique of Woodhouse & Dahlen (1978). Instead, we obtain the matrix elements by a straightforward perturbation of the equations of motion. Of course, there are matrix elements in the solar case that are not required in the terrestrial case. These terms have motivated the present paper. Likewise, there are terms due to the rigidity, elastic anisotropy, and deviatoric stress of the Earth that are not necessary in the solar case. The matrix elements derived in this paper and those derived in Woodhouse (1980) that are common to the Earth and the Sun are in agreement (see the discussion at the end of Appendix E*d*).

The Wigner–Eckart theorem (eq. [5.4.1] of Edmonds 1960) states that the general matrix element of a tensor perturbation operator can be expanded in a series of

Wigner $3j$ symbols whose coefficients of expansion are independent of m and m'. Thus the general matrix element can be written

$$H^{m'm}_{n'n,\,l'l} = (-1)^{m'} \sum_{s=0}^{\infty} \sum_{t=-s}^{s} \begin{pmatrix} s & l' & l \\ t & -m' & m \end{pmatrix} (n'l' \,\|\delta H^t_s\|\, nl). \tag{77}$$

The coefficients appearing in the expansion are called the reduced or double-bar matrix elements and are radial integrals of the form

$$(n'l' \,\|\delta H^t_s\|\, nl) = \int_0^{R_\odot} \mathscr{G}^{(n'l',\,nl)}_s(r) \cdot \delta \boldsymbol{m}^t_s(r)\, r^2 \, \mathrm{d}r, \tag{78}$$

where $\mathscr{G}^{(n'l',\,nl)}_s$ is a vector of kernels that depend on the modal eigenfunctions and the SNRNMAIS model parameters, and $\delta \boldsymbol{m}^t_s(r)$ is a vector of expansion coefficients that fully prescribes the aspherical scalar and vector fields of the non-SNRNMAIS solar model. The term on the right-hand side of equation (77) with $(s=0, t=0)$ corresponds to the spherical part of the perturbation and vanishes for a model with purely aspherical perturbations. In the following we obtain the reduced matrix elements for convective flow and for aspherical perturbations in the elastic-gravitational variables. We present formal expressions for the various contributions to the general matrix element in terms of vector-integral operations. In Appendixes D and E we manipulate these expressions to eliminate undesirable derivatives of SNRNMAIS model parameters and aspherical expansion coefficients. The means of expressing $H^{m'm}_{n'n,\,l'l}$ as in equation (77) is the subject of Appendix C.

For purposes of presentation, it is useful to separate $H^{m'm}_{n'n,\,l'l}$ into inertial (or kinetic energy) $T^{m'm}_{n'n,\,l'l}$ and elastic-gravitational (or potential energy) $V^{m'm}_{n'n,\,l'l}$ contributions. Thus, we define

$$H^{m'm}_{n'n,\,l'l} = T^{m'm}_{n'n,\,l'l} - V^{m'm}_{n'n,\,l'l}, \tag{79}$$

where by equation (76)

$$T^{m'm}_{n'n,\,l'l} = \int \rho_0\, \boldsymbol{s}^*_{k'} \cdot \boldsymbol{T}(\boldsymbol{s}_k)\, \mathrm{d}^3\boldsymbol{r}, \tag{80}$$

$$V^{m'm}_{n'n,\,l'l} = \int \boldsymbol{s}^*_{k'} \cdot (\delta \mathscr{L}_0(\boldsymbol{s}_k) + \delta\rho_0\, \omega^2_{\text{ref}}\, \boldsymbol{s}_k)\, \mathrm{d}^3\boldsymbol{r}. \tag{81}$$

In §§6a, b we decompose $T^{m'm}_{n'n,\,l'l}$ and $V^{m'm}_{n'n,\,l'l}$ into physically meaningful units. In §6c we assemble the general matrix elements for all scalar and vector perturbations and present their form explicitly in terms of the scalar eigenfunctions of the SNRNMAIS modes and the expansion coefficients that fully prescribe the asphericities (see §3). The final result is given by equation (90).

(a) *The inertial contribution to the general matrix element*

It is instructive to decompose $T^{m'm}_{n'n,\,l'l}$ in the form

$$T^{m'm}_{n'n,\,l'l} = B^{m'm}_{n'n,\,l'l} + C^{m'm}_{n'n,\,l'l}, \tag{82}$$

where by equations (B 22)–(B 24):

$$B^{m'm}_{n'n,\,l'l} = \int \rho_0\, \boldsymbol{s}^*_{k'} \cdot 2\mathrm{i}\omega \boldsymbol{\Omega} \times \boldsymbol{s}_k\, \mathrm{d}^3\boldsymbol{r}, \tag{83}$$

$$C^{m'm}_{n'n,\,l'l} = \int \rho_0\, \boldsymbol{s}^*_{k'} \cdot 2\mathrm{i}\omega \boldsymbol{u}_0 \cdot \nabla \boldsymbol{s}_k\, \mathrm{d}^3\boldsymbol{r}. \tag{84}$$

Clearly $B^{m'm}_{n'n,l'l}$ is the general matrix element arising from the Coriolis force, and $C^{m'm}_{n'n,l'l}$ is due to the advection of modes by differential rotation and the convective flow field with respect to the corotating frame. The matrix elements are presented in terms of the scalar eigenfunctions in Appendix D.

(b) *The elastic-gravitational contribution to the general matrix element*

To calculate $V^{m'm}_{n'n,l'l}$, we substitute $\delta\mathscr{L}_0$ from equation (B 25) into equation (81). Upon separating terms in $\delta\kappa_0$, $\delta\rho_0$, and $\delta\Phi_0$, we find that $V^{m'm}_{n'n,l'l}$ can be written

$$V^{m'm}_{n'n,l'l} = K^{m'm}_{n'n,l'l} + R^{m'm}_{n'n,l'l} + P^{m'm}_{n'n,l'l}, \tag{85}$$

where $K^{m'm}_{n'n,l'l}$, $R^{m'm}_{n'n,l'l}$, and $P^{m'm}_{n'n,l'l}$ denote, respectively, the general matrix elements for the perturbed bulk modulus, perturbed density, and perturbed gravitational potential (and its gradient). They are given by

$$K^{m'm}_{n'n,l'l} = \int \boldsymbol{s}^*_{k'}\cdot\nabla(\delta\kappa_0\,\nabla\cdot\boldsymbol{s}_k)\,\mathrm{d}^3\boldsymbol{r}, \tag{86}$$

$$\begin{aligned} R^{m'm}_{n'n,l'l} = {} & \int \delta\rho_0\{\omega^2_{\mathrm{ref}}\,\boldsymbol{s}^*_{k'}\cdot\boldsymbol{s}_k + \boldsymbol{s}^*_{k'}\cdot\nabla\phi_0(\nabla\cdot\boldsymbol{s}_k) - \boldsymbol{s}^*_{k'}\cdot\nabla\delta\phi(\boldsymbol{s}_k) - \boldsymbol{s}^*_{k'}\cdot[\boldsymbol{s}_k\cdot\nabla(\nabla\phi_0)]\}\,\mathrm{d}^3\boldsymbol{r} \\ & - \int \boldsymbol{s}^*_{k'}\cdot\nabla(\delta\rho_0\,\boldsymbol{s}_k\cdot\nabla\phi_0)\,\mathrm{d}^3\boldsymbol{r} + \int \boldsymbol{s}^*_{k'}\cdot[\boldsymbol{s}_k\cdot\nabla(\delta\rho_0\,\nabla\phi_0)]\,\mathrm{d}^3\boldsymbol{r}, \end{aligned} \tag{87}$$

$$P^{m'm}_{n'n,l'l} = \int \boldsymbol{s}^*_{k'}\cdot\{\rho_0(\nabla\cdot\boldsymbol{s}_k)\,\nabla\delta\Phi_0 - \nabla(\rho_0\,\boldsymbol{s}_k\cdot\nabla\delta\Phi_0) + \boldsymbol{s}_k\cdot\nabla(\rho_0\,\nabla\delta\Phi_0) - \rho_0\,\boldsymbol{s}_k\cdot\nabla(\nabla\delta\Phi_0)\}\,\mathrm{d}^3\boldsymbol{r}. \tag{88}$$

These matrix elements are reduced and simplified in Appendix E.

(c) *The final form of the general matrix element*

We now combine the contributions from the inertial and elastic-gravitational matrix elements to obtain the final expression for the general matrix element. By using equations (79), (82) and (85), the general matrix element can be written

$$H^{m'm}_{n'n,l'l} = B^{m'm}_{n'n,l'l} + C^{m'm}_{n'n,l'l} - K^{m'm}_{n'n,l'l} - R^{m'm}_{n'n,l'l} - P^{m'm}_{n'n,l'l}. \tag{89}$$

Combining equations (85), (D 1), (D 17), and equation (E 25), we obtain

$$\begin{aligned} H^{m'm}_{n'n,l'l} = {} & \delta_{ll'}\,\delta_{mm'}\left[2m\Omega\omega_{\mathrm{ref}}\int_0^{R_\odot}\rho_0\,C(r)\,r^2\,\mathrm{d}r + \tfrac{2}{3}\Omega^2\left(\delta_{nn'} - l(l+1)\int_0^{R_\odot}\rho_0\,C(r)\,r^2\,\mathrm{d}r\right)\right] \\ & + E^m_l\int_0^{R_\odot}E(r)\,r^2\,\mathrm{d}r + 4\pi(-1)^{m'}\gamma_{l'}\gamma_l\sum_{s=0}^{\infty}\gamma_s\sum_{t=-s}^{s}\begin{pmatrix} l' & s & l \\ -m' & t & m\end{pmatrix}\int_0^{R_\odot}\{\delta\kappa^t_s(r)\,K_s(r) \\ & + \delta\rho^t_s(r)\,R^{(2)}_s(r) + 2\omega_{\mathrm{ref}}\,\rho_0(\mathrm{i}u^t_s(r)\,R_s(r) + \mathrm{i}v^t_s(r)\,H_s(r) + w^t_s(r)\,T_s(r))\}\,r^2\,\mathrm{d}r, \end{aligned} \tag{90}$$

where the constant factors are given by

$$\gamma_s = \sqrt{((2s+1)/4\pi)}, \tag{91}$$

$$E^m_l = \delta_{m'm}(T_{lm}\,\delta_{l'l} + \tfrac{3}{2}S_{lm}\,S_{l'+1,m}\,\delta_{ll'+2} + \tfrac{3}{2}S_{l'm}\,S_{l+1,m}\,\delta_{l'l+2}), \tag{92}$$

$$S_{lm} = \left[\frac{(l+m)(l-m)}{(2l+1)(2l-1)}\right]^{\frac{1}{2}}, \tag{93}$$

$$T_{lm} = \frac{l(l+1)-3m^2}{(2l-1)(2l+3)}, \tag{94}$$

and the sensitivity kernels are given by

$$C(r) = (UV' + U'V + VV'), \tag{95}$$

$$R_s(r) = \tfrac{1}{2}(U'\dot{U} - \dot{U}'U)B^{(0)+}_{l'sl} + \tfrac{1}{2}(V'\dot{V} - V'V)B^{(1)+}_{l'sl} \tag{96}$$

$$H_s(r) = \frac{1}{2r}[l(l+1) - l'(l'+1)]\,[(U'U)\,B^{(0)+}_{l'sl} + (V'V)\,B^{(1)+}_{l'sl}] + r^{-1}V'UB^{(1)+}_{l'ls} - r^{-1}U'VB^{(1)+}_{ll's}, \tag{97}$$

$$T_s(r) = r^{-1}\{U'V + V'U - U'U - \tfrac{1}{2}V'V[l(l+1) + l'(l'+1) - s(s+1)]\}\,B^{(1)-}_{l'sl}, \tag{98}$$

$$E(r) = \tfrac{2}{3}\epsilon(r)\,[\kappa_0(\bar{K}(r) - (\eta+1)\,\tilde{K}(r)) + \rho_0(\bar{R}(r) - (\eta+3)\,\tilde{R}(r))], \tag{99}$$

$$K_s(r) = (\dot{U}' + F')\,(\dot{U} + F)\,B^{(0)+}_{l'sl}, \tag{100}$$

$$R^{(2)}_s(r) = R^{(1)}_s(r) + \frac{4\pi G}{2s+1}\left\{r^2 \int_r^{R_\odot} r^{-s}[(s+1)\,G^{(2)}_s(r) - rG^{(1)}_s(r)]\,\mathrm{d}r - r^{-s-1}\int_0^r r^{s+1}[sG^{(2)}_s(r) + rG^{(1)}_s(r)]\,\mathrm{d}r\right\}, \tag{101}$$

$$R^{(1)}_s(r) = [-\omega^2_{\mathrm{ref}}\,VV' + r^{-1}(\delta\phi' V + \delta\phi V') + \tfrac{1}{2}g_0\,r^{-1}(U'V + V'U)]\,B^{(1)+}_{l'sl} + [8\pi G\rho_0\,UU' + \delta\dot{\phi}' + \delta\dot{\phi}U' - \omega^2_{\mathrm{ref}}\,UU' - \tfrac{1}{2}g_0(4r^{-1}UU' + U'F + UF')]\,B^{(0)+}_{l'sl}, \tag{102}$$

$$G^{(1)}_s(r) = \tfrac{1}{2}\rho_0\,r^{-1}(U\dot{V}' + r^{-1}UV' - \dot{U}V' - 2FV')\,B^{(1)+}_{l'ls} + \tfrac{1}{2}\rho_0\,r^{-1}(U'\dot{V} + r^{-1}U'V - \dot{U}'V - 2F'V)\,B^{(1)+}_{ll's} + \rho_0\,r^{-2}UU's(s+1)\,B^{(0)+}_{l'sl}, \tag{103}$$

$$G^{(2)}_s(r) = \tfrac{1}{2}\rho_0\,r^{-1}UV'B^{(1)+}_{l'ls} + \tfrac{1}{2}\rho_0\,r^{-1}U'VB^{(1)+}_{ll's} - \rho_0(F'U + U'F)\,B^{(0)+}_{l'sl}, \tag{104}$$

$$\bar{K}(r) = -(\dot{U} + F)\,(\dot{U}' + \tfrac{1}{2}(l'(l'+1) - l(l+1) + 6)\,r^{-1}V') - (\dot{U}' + F')\,(\dot{U} + \tfrac{1}{2}(l(l+1) - l'(l'+1) + 6)\,r^{-1}V), \tag{105}$$

$$\begin{aligned}\bar{R}(r) = {} & F(r\delta\dot{\phi}' + 4\pi G\rho_0\,rU' + g_0\,U') \\ & + \tfrac{1}{2}(l'(l'+1) - l(l+1) + 6)\,UV'(\omega^2_{\mathrm{ref}} - r^{-1}g_0) + 3r^{-1}g_0\,UU' \\ & + r^{-1}\delta\phi'[\tfrac{1}{2}(l'(l'+1) + l(l+1) - 6)\,V - l'(l'+1)\,U] \\ & + F'(r\delta\dot{\phi} + 4\pi G\rho_0\,rU + g_0\,U) + \tfrac{1}{2}(6 + l(l+1) - l'(l'+1))\,U'V(\omega^2_{\mathrm{ref}} - r^{-1}g_0) \\ & + 3r^{-1}g_0\,U'U + r^{-1}\delta\phi[\tfrac{1}{2}(l(l+1) + l'(l'+1) - 6)\,V' - l(l+1)\,U'],\end{aligned} \tag{106}$$

$$\tilde{K}(r) = \tfrac{1}{2}(\dot{U} + F)\,(-\dot{U}' + F' + (l'(l'+1) - l(l+1) + 6)\,r^{-1}V') + \tfrac{1}{2}(\dot{U}' + F')\,(-\dot{U} + F + (l(l+1) - l'(l'+1) + 6)\,r^{-1}V), \tag{107}$$

$$\begin{aligned}\tilde{R}(r) = {} & \tfrac{1}{4}(l(l+1) + l'(l'+1) - 6)\,(2r^{-1}V'\delta\phi - \omega^2_{\mathrm{ref}}\,VV') \\ & + \tfrac{1}{2}U'[2\delta\dot{\phi} + 8\pi G\rho_0\,U - \omega^2_{\mathrm{ref}}\,U - (l(l+1) \\ & - l'(l'+1) + 6)\,g_0\,r^{-1}V] + \tfrac{1}{4}(l(l+1) + l'(l'+1) - 6)\,(2r^{-1}V\,\delta\phi' - \omega^2_{\mathrm{ref}}\,VV') \\ & + \tfrac{1}{2}U[2\delta\dot{\phi}' + 8\pi G\rho_0\,U' - \omega^2_{\mathrm{ref}}\,U' - (l'(l'+1) - l(l+1) + 6)\,g_0\,r^{-1}V'],\end{aligned} \tag{108}$$

$$F(r) = r^{-1}(2U - l(l+1)\,V), \tag{108}$$

$$F'(r) = r^{-1}(2U' - l'(l'+1)\,V'), \tag{110}$$

where $C(r)$ is the Coriolis kernel, $E(r)$ is the elipticity kernel, $K_s(r)$ is the bulk modulus perturbation kernel, $R^{(2)}_s(r)$ is the density perturbation kernel, $R_s(r)$ and $H_s(r)$ are the poloidal flow kernels, and $T_s(r)$ is the toroidal flow kernel. The velocity field expansion coefficients $u^t_s(r)$, $v^t_s(r)$, and $w^t_s(r)$ are defined by equation (23), and the expansion coefficients for the elastic-gravitational perturbations $\delta\kappa^t_s(r)$ and $\delta\rho^t_s(r)$ are defined, respectively, by equations (35) and (36). The factor η and the ellipticity $\epsilon(r)$ are defined, respectively, by eqs (A 10) and (A 11) of Woodhouse & Dahlen (1978). The coefficients $B_{l'sl}^{(N)\pm}$ are defined by equation (C 44). The scalar eigenfunctions of the SNRNMAIS modes are given by $\delta\phi$, $\delta\dot{\phi}$, U and V. The kernels corresponding to perturbations in the gravitational potential $\delta\phi_0$ and its radial derivative $\delta\dot{\phi}_0$ have been incorporated into the density perturbation kernel (see eqs (A 48)–(A 50) in Woodhouse 1980). The Wigner $3j$ symbol in equation (90) can be computed numerically by using the algorithm of Zare (1989).

The first integral in equation (90) represents the effect of the Coriolis force. The second term models the effect of the spherical part of the centripetal acceleration. This term does not satisfy the diagonal sum rule derived in §7*d*. The reason for this is that the reference model is the SNRNMAIS model rather than a rotating model that includes a spherically symmetric body force distribution representing the spherical average of the centripetal force. The third integral in equation (90) represents the effect of elipticity and the latitude-dependent part of the centripetal acceleration. To see that this term models the latitude dependence of the centripetal acceleration, compare $E(r)$ to eqs (97) and (A 14) of Woodhouse & Dahlen (1978). The fourth integral in equation (90) models the effect of convective flow and aspherical perturbations to the elastic-gravitational variables.

We note that by perturbing the sound speed c_0 (where $c_0 = \surd(\kappa_0/\rho_0)$), the elastic-gravitational kernels can be transformed from perturbations in κ_0 and ρ_0 into perturbations in c_0 and ρ_0:

$$\delta\kappa_s^t(r)\,K_s(r) + \delta\rho_s^t(r)\,R_s^{(2)}(r) \rightarrow \delta c_s^t(r)\,[2\rho_0\,c_0\,K_s(r)] + \delta\rho_s^t(r)\,[R_s^{(2)}(r) + c_0^2\,K_s(r)], \quad (111)$$

where $\delta c_s^t(r)$ is the spherical harmonic expansion coefficient for the (s,t) component of the aspherical perturbation to the sound speed and is defined by the relation

$$\delta c_0(r,\theta,\phi) = \sum_{s=0}^{\infty}\sum_{t=-s}^{s} \delta c_s^t(r)\,Y_s^t(\theta,\phi). \quad (112)$$

This expansion is for all aspherical perturbations of c_0 independent of effects induced by rigid rotation; the ellipticity kernel need not be adjusted.

7. Properties of the supermatrix

In §6, we presented expressions that can be used to compute the general matrix elements that compose the supermatrix $\boldsymbol{Z}$ given the eigenfunctions of the SNRNMAIS model and the vector spherical harmonic representation of a stationary convective velocity field and asphericities in the elastic-gravitational variables. In this section, we discuss properties of the supermatrix. Much of our discussion focuses on considering which SNRNMAIS modes can couple due to a given convective flow field or asphericity. We say two SNRNMAIS modes k and k' couple if their eigenfunctions contribute to the linear combination representing an eigenfunction of a non-SNRNMAIS model. That is, they couple if the expansion coefficients a_k^j and $a_{k'}^j$ are non-zero in equation (6). The strength of coupling is dependent on factors we will discuss below. In the remainder of this section we discuss (1) how to construct the supermatrix, (2) a new rationale for the quasi-degeneracy condition based on a consideration of the properties of eigensystems, (3) the angular selection rules that govern which SNRNMAIS modes can couple in the presence of a specified convective flow field or asphericity in the elastic-gravitational variables, and (4) other considerations that affect coupling strengths, notably the characteristics of the radial kernels that compose the general matrix elements. These discussions are the subjects of §§7*a–c*, respectively. In addition, in §7*d* we discuss a property of the average frequency of modes composing a split multiplet that is also a direct consequence of the Wigner–Eckart theorem. This property is known as the diagonal sum rule.

(*a*) *Constructing the supermatrix*

Again, let $k = (n, l, m)$ and $k' = (n', l', m')$. Equation (67) shows that the supermatrix $\boldsymbol{Z}$ is the sum of two matrices, a diagonal matrix with entries $(\omega_{nl}^2 - \omega_{\text{ref}}^2)$ and general matrices $\boldsymbol{H}_{n'n,l'l}$. Let us consider the construction of the supermatrix for a narrow frequency band around a given multiplet ${}_nS_l$ with degenerate frequency ω_{ref} in accordance with the quasi-degeneracy condition. We include the modes from all multiplets with eigenfunctions in K, the eigenspace of all modes whose degenerate eigenfrequencies lie in the specified band. With a notational shift defined directly below, the supermatrix can be written:

$$\boldsymbol{Z} = \begin{bmatrix} \ddots & & & \\ & (\omega_{-1}^2 - \omega_{\text{ref}}^2)\,\boldsymbol{I}_{-1-1} & \boldsymbol{0} & \\ & & (\omega_1^2 - \omega_{\text{ref}}^2)\,\boldsymbol{I}_{11} & \\ & & & \ddots \end{bmatrix} + \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \boldsymbol{H}_{-1-1} & \boldsymbol{H}_{-10} & \boldsymbol{H}_{-11} & \cdots \\ \cdots & \boldsymbol{H}_{0-1} & \boldsymbol{H}_{00} & \boldsymbol{H}_{01} & \cdots \\ \cdots & \boldsymbol{H}_{1-1} & \boldsymbol{H}_{10} & \boldsymbol{H}_{11} & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}, \tag{113}$$

where the normalization factors N_k have been absorbed into the general matrices. In this notation, the matrix $\boldsymbol{H}_{00}$ is the splitting matrix governing the self-coupling among the SNRNMAIS modes of ${}_nS_l$. That is, returning momentarily to our original notation for the matrix elements, it includes the general matrix elements $H_{nn,ll}^{m'm}$ as follows:

$$\boldsymbol{H}_{00} = \begin{bmatrix} & \vdots & \vdots & & \\ \cdots & H_{nn,ll}^{-1-1} & H_{nn,ll}^{-10} & H_{nn,ll}^{-11} & \cdots \\ \cdots & H_{nn,ll}^{0-1} & H_{nn,ll}^{00} & H_{nn,ll}^{01} & \cdots \\ \cdots & H_{nn,ll}^{1-1} & H_{nn,ll}^{10} & H_{nn,ll}^{11} & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{114}$$

The azimuthal-order superscripts in equation (114) determine the location of the matrix element within the diagonal block splitting matrix. All matrices on the diagonal of the right-most term in equation (113) are square splitting matrices governing the self-coupling among the SNRNMAIS modes for each multiplet in K. The subscripts denoting each diagonal block $\boldsymbol{H}_{ii}$ are positive or negative depending on whether the degenerate frequency of the multiplet ω_{nl} is greater than or less than ω_{ref}. The off-diagonal coupling blocks $\boldsymbol{H}_{ij}$ represent the coupling terms between multiplets identified with the i and j subscripts. These blocks are composed of the general matrix elements $H_{n'n,l'l}^{m'm}$ and are not square since $-l' \leqslant m' \leqslant l'$, but $-l \leqslant m \leqslant l$. Consider, for example, the off-diagonal block $\boldsymbol{H}_{-10}$ where the multiplet ${}_{n'}S_{l'}$

has a lower degenerate frequency than ${}_nS_l$. Again, returning again temporarily to our original notation for the matrix elements, this non-square matrix would appear as follows:

$$\boldsymbol{H}_{-10} = \begin{bmatrix} & \vdots & \vdots & & \\ \cdots & H^{-1-1}_{n'n,l'l} & H^{-10}_{n'n,l'l} & H^{-11}_{n'n,l'l} & \cdots \\ \cdots & H^{0-1}_{n'n,l'l} & H^{00}_{n'n,l'l} & H^{01}_{n'n,l'l} & \cdots \\ \cdots & H^{1-1}_{n'n,l'l} & H^{10}_{n'n,l'l} & H^{11}_{n'n,l'l} & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{115}$$

Under degenerate perturbation theory only self-coupling is allowed, so the supermatrix would be block-diagonal, being composed only of the splitting matrices $\boldsymbol{H}_{ii}$. In this case, no coupling could take place between multiplets, only within each individual multiplet. This is a useful way to envision how degenerate perturbation theory approximates quasi-degenerate perturbation theory. We will investigate its accuracy numerically in a later paper.

The hermiticity of the supermatrix (i.e. $\boldsymbol{Z} = \boldsymbol{Z}^{\dagger}$, where the superscript $\dagger$ denotes the complex conjugate transpose) is tantamount to the property that the real general matrix elements $H^{m'm}_{n'n,l'l}$ are invariant under the swapping of the primed and unprimed indices and the complex elements are skew-symmetric. An inspection of equation (90) reveals that $H^{m'm}_{n'n,l'l}$ satisfies these conditions and, therefore, that the supermatrix, composed of general matrix elements, is hermitian. If the anelastic condition on the flow field had not been applied, then the complex matrix elements composed of the purely imaginary poloidal flow kernels R_s and H_s would not have been skew-symmetric (as can be seen from (C 32) and (C 33)) and the supermatrix would not be hermitian. The hermiticity of $\boldsymbol{Z}$ guarantees that the modal frequencies computed from the theory presented here are purely real.

(*b*) *The quasi-degeneracy condition reconsidered*

In §5, we motivated the quasi-degeneracy condition by referring to the general property of linear oscillators, that they only couple strongly if their unperturbed frequencies are nearly degenerate. Understanding that modal frequencies are eigenvalues of the supermatrix now allows us to reformulate the condition more rigorously, basing it on a property of eigensystems. Consider the following symmetric 2×2 matrix:

$$\mathscr{A} = \begin{bmatrix} -\tfrac{1}{2}\Delta & c \\ c & \tfrac{1}{2}\Delta \end{bmatrix}. \tag{116}$$

By analogy with the supermatrix above, the off-diagonal elements c represent coupling terms. In the absence of the coupling terms, the eigenvalues of $\mathscr{A}$ are $\pm\tfrac{1}{2}\Delta$. Assuming $\tfrac{1}{2}\Delta > |c|$ (i.e. $\Delta \neq 0$), the perturbation to the positive eigenvalue caused by the off-diagonal elements is approximately

$$\nu = c^2/\Delta. \tag{117}$$

That is, the positive eigenvalue of the matrix $\mathscr{A}$ is approximately $\tfrac{1}{2}\Delta + \nu$. This formula continues to hold for any otherwise diagonal $n \times n$ matrix with a single pair of symmetric off-diagonal perturbations. For a general $n \times n$ symmetric matrix, the eigenvalues are related nonlinearly to any pair of off-diagonal perturbations. However, ν remains a good approximate measure of the effect of any single pair of off-diagonal elements and we call it the coupling strength coefficient. It is a measure of the potential of any off-diagonal pair of coupling terms to affect the eigenfrequency.

The coupling strength coefficient decreases linearly with the difference between diagonal elements linked by the off-diagonal pair of coupling terms. Applied to the supermatrix, this means that ν decreases linearly as the difference between the eigenfrequencies of two modes of the axisymmetric model. This observation is the basis of the quasi-degeneracy condition. Off-diagonal perturbations to the supermatrix effectively perturb eigenfrequencies only if the diagonal elements are nearly degenerate.

(c) *Selection rules*

We discuss in detail the selection rules that govern coupling of SNRNMAIS modes through convective flow and elastic-gravitational asphericities. As stated in §5*a*, the satisfaction of the quasi-degeneracy condition alone does not guarantee that an eigenfunction of mode k will be included in K. The mode must also satisfy certain selection rules, the result of which would guarantee that the mode can couple with at least one other mode in K. In terms of the supermatrix, this means that each column of $\boldsymbol{Z}$ would have at least two non-zero elements.

The Wigner–Eckart theorem (equation (77)) guarantees that the general matrix element $H^{m'm}_{n'n,l'l}$ can be decomposed into a sum over products of Wigner 3*j* symbols and reduced matrix elements $(n'l'\,\|\delta H^t_s\|\,nl)$. As discussed in Appendix C, the Wigner 3*j* symbol is proportional to the integral of the product of three generalized spherical harmonics over the unit sphere. Since such surface integrations are insensitive to the radial orders of the modes, any selection rules on n and n' must be inferred from the reduced matrix elements. In §§7*c*(i) and (ii), we obtain, respectively, selection rules that follow from the Wigner 3*j* symbols and from the reduced matrix elements. In each of these subsections we first consider selection rules governing the general case of cross-coupling. We then apply these rules to the special case of self-coupling, and list these selection rules separately, denoting them with the subscript sc.

(i) *Selection rules from the Wigner 3j symbols*

We state here two general selection rules resulting from properties of the Wigner 3*j* symbols that apply both to coupling caused by convective flows and by elastic-gravitational asphericities. The range in l and l' over which the SNRNMAIS modes k and k' can couple depends on the range in harmonic degree s of the spherical and vector spherical harmonic basis functions that represent convective flow and structural asphericities. From the property of Wigner 3*j* symbols in equation (C 43), we find that the harmonic degrees l, l', and s must satisfy the following triangle inequalities:

$$\text{selection rule 1}\quad \left.\begin{aligned} |l'-s| &\leqslant l,\\ |s-l| &\leqslant l',\\ |l-l'| &\leqslant s.\end{aligned}\right\} \tag{118}$$

For example, consider coupling between two modes whose harmonic degrees differ by 2; say, $l' = l+2$. Then the application of selection rule 1 guarantees that $2 \leqslant s \leqslant 2l+2$. We consider this example further in a discussion of differential rotation in §8. In the special case of self-coupling ($n' = n$, $l' = l$), selection rule 1 reduces to a particularly simple form:

$$\text{selection rule } 1_{\text{sc}}\quad 0 \leqslant s \leqslant 2l. \tag{119}$$

That is, under self-coupling a mode is only sensitive to convective flow and aspherical structure with harmonic degree up through degree $2l$.

The second general selection rule also follows from equation (C 43). It states that modes with azimuthal order m and m' will couple only if there exists a component of convective flow or aspherical structure with azimuthal order t such that

$$\text{selection rule 2 and } 2_{\text{sc}} \quad -m'+t+m=0. \tag{120}$$

Selection rule 2 holds in identical form in the self-coupling case. Thus, when the azimuthal order of convection or aspherical structure $t=0, m'=m$; when $t=\pm 1$, $m'=m\pm 1$; when $t=\pm 2$, $m'=m\pm 2$; and so forth. Thus, as mentioned in §7*a*, the contribution to a splitting matrix by axisymmetric structure is on the diagonal, the contribution by $t=\pm 1$ convection or aspherical structure is on the first off-diagonal, and so forth.

(ii) *Selection rules from the reduced matrix elements*

We consider here properties that can be deduced from the reduced matrix elements for convective flow and elastic-gravitational asphericities, the integral kernels of which are listed in §6*c*. The reduced matrix elements can be constructed by comparing the expression for the general matrix element in equation (90) with the form given by the Wigner–Eckart theorem in equation (77). The reduced matrix elements for convective flow are proportional to the poloidal flow kernels R_s and H_s (equations (96) and (97)), and to the toroidal flow kernel T_s (equation (98)). The reduced matrix elements of the poloidal flow terms are for an anelastic velocity field; selection rules resulting from the reduced matrix elements for a general velocity field would be different. The reduced matrix elements for the elastic-gravitational asphericities are proportional to the kernel K_s (equation (100)) and to the kernel $R_s^{(2)}$ (equation (101)). The kernels R_s, H_s, K_s and $R_s^{(2)}$ are proportional to either $B_{l'sl}^{(0)+}$ or $B_{l'sl}^{(1)+}$ (defined by (C 44)). These coefficients are non-zero only if the sum $l'+l+s$ is even. Thus the SNRNMAIS modes k and k' cannot couple through poloidal flows or through perturbations to the density and adiabatic bulk modulus unless the sum $l'+l+s$ is even. The toroidal flow kernel T_s is proportional to $B_{l'sl}^{(1)-}$. This coefficient vanishes unless the sum $l'+l+s$ is odd. Thus, the SNRNMAIS modes k and k' cannot couple through a toroidal flow unless the sum $l'+l+s$ is odd. These observations are encapsulated in selection rule 3:

$$\text{selection rule 3} \quad \left.\begin{array}{c} R_s=H_s=K_s=R_s^{(2)}=0 \quad \text{if} \quad l'+l+s \text{ is odd,} \\ T_s=0 \quad \text{if} \quad l'+l+s \text{ is even.} \end{array}\right\} \tag{121}$$

In the case of self-coupling, selection rule 3 simplifies considerably:

$$\text{selection rule } 3_{\text{sc}} \quad \left.\begin{array}{c} R_s=H_s=0, \\ T_s=0 \quad \text{if} \quad s \text{ is even,} \\ K_s=R_s^{(2)}=0 \quad \text{if} \quad s \text{ is odd.} \end{array}\right\} \tag{122}$$

By selection rule 3_{sc}, the poloidal flow kernels R_s and H_s vanish for self-coupling. Thus under the self-coupling approximation or within degenerate perturbation theory, an anelastic, poloidal velocity field does not couple SNRNMAIS modes. In addition, only the odd degree s component of toroidal flows couples SNRNMAIS modes. In summary, only odd-degree toroidal flows and even degree elastic-gravitational asphericities couple or split SNRNMAIS modes within self-coupling.

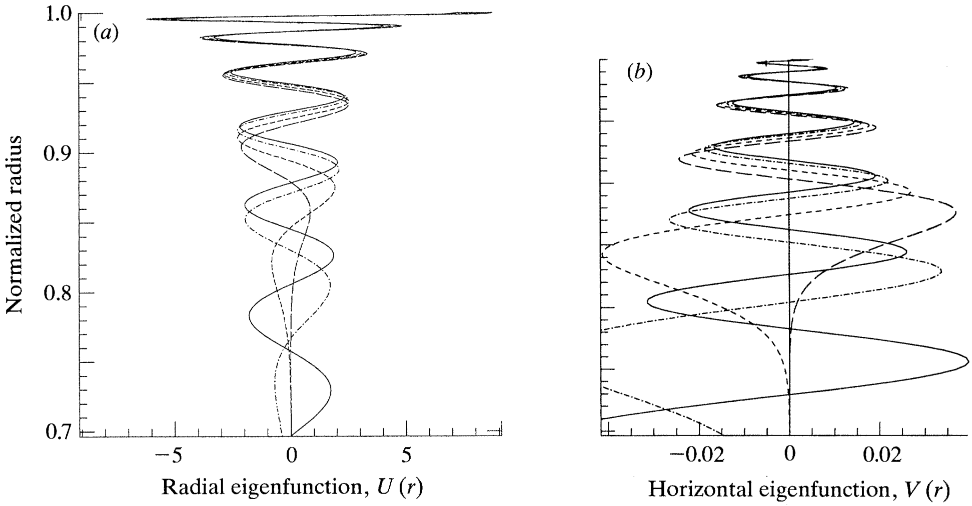

Figure 3. (*a*) The quantity $\surd(\rho_0 U^2)$ plotted as a function of radius for the multiplets ${}_{14}S_{30}$ (3.25013 mHz), ${}_{11}S_{49}$ (3.25001 mHz), ${}_{9}S_{70}$ (3.24938 mHz), and ${}_{8}S_{84}$ (3.24786 mHz). The shapes of the eigenfunctions U and V (see (*b*)) in the upper region of the convection zone depend primarily on the degenerate frequencies of the multiplets. Eigenfunction shapes at a constant frequency are similar even for widely differing harmonic degree l. ————, $n = 8$, $l = 84$; – – – –, $n = 9$, $l = 70$; -·—·-·—·-·—, $n = 11$, $l = 49$; ——, $n = 14$, $l = 30$. (*b*) The quantity $\surd(\rho_0 V^2)$ plotted for the same multiplets as in (*a*). This plot illustrates that the eigenfunctions V in the upper region of the convection zone are strong functions of their degenerate frequencies.

Under self-coupling, the radial poloidal flow kernel R_s in equation (96) vanishes identically. However, the radial poloidal flow kernel also vanishes approximately for near surface flows for cross-coupling since, as can be seen from figure 3*a* and *b*, nearly degenerate modes have similar eigenfunctions U and V near the solar surface. Thus $(U'\dot{U}-\dot{U}'U) \approx (V'\dot{V}-\dot{V}'V) \sim 0$ and we can list the following approximate selection rule:

$$\text{approximate selection rule 4} \quad R_s \approx 0 \text{ for near surface flows.} \tag{123}$$

Thus the radial component of flow does not strongly affect acoustic waves strongly near the solar surface.

(*d*) *The diagonal sum rule*

In this section we derive the diagonal sum rule and the super-diagonal sum rule which state that the average frequency of a split multiplet or set of split, coupling multiplets remains unchanged by a perturbation in the model such as convective flow or asphericities in the elastic-gravitational variables. The diagonal sum rule applies in the self-coupling approximation, and asserts that the average frequency of a split multiplet is simply the degenerate frequency of the multiplet. It was first proven by Gilbert (1971). The super-diagonal sum rule extends the diagonal sum rule to coupling multiplets. Although the former is a special case of the latter, we find that for purposes of presentation, it is advantageous to derive the diagonal sum rule first.

First, we consider self-coupling. We begin by using the well known fact that traces of matrices with the same characteristic polynomials are identical. It follows that similar matrices have the same trace. Let the matrix $\boldsymbol{A}$ diagonalize the splitting matrix $\boldsymbol{H}_{nn,ll}$; i.e.

$$\boldsymbol{W} = \boldsymbol{A}^{-1}\boldsymbol{H}_{nn,ll}\boldsymbol{A}, \tag{124}$$

19-2

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

where $\boldsymbol{W}$ is the diagonal matrix containing the $(2l+1)$ eigenvalues $2\omega_{nl}\,\delta\omega_j$ of $\boldsymbol{H}_{nn,ll}$. Therefore, the trace of $\boldsymbol{H}_{nn,ll}$ is given by

$$\operatorname{tr}\boldsymbol{H}_{nn,ll} = \sum_{m=-l}^{l} H^{mm}_{nn,ll} = 2\omega_{nl}\sum_{j=1}^{2l+1}\delta\omega_j. \tag{125}$$

We use the Wigner–Eckart theorem to perform the sum in equation (125). The diagonal elements of the splitting matrix can be obtained from equation (77) by setting $m' = m$. Using selection rule 2, we find the only non-zero Wigner $3j$ symbol in the Wigner–Eckart decomposition for the case $m' = m$ is given by the term with $t = 0$. Thus, the trace of $\boldsymbol{H}_{nn,ll}$ can be written

$$\operatorname{tr}\boldsymbol{H}_{nn,ll} = \sum_{s=0}^{\infty}(nl\,\|\delta H^0_s\|\,nl)\left[\sum_{m=-l}^{l}(-1)^m\begin{pmatrix} s & l & l\\ 0 & -m & m\end{pmatrix}\right]. \tag{126}$$

It is a property of Wigner $3j$ symbols that

$$\sum_{m=-l}^{l}(-1)^m\begin{pmatrix} s & l & l\\ 0 & -m & m\end{pmatrix} = 0 \quad \text{for} \quad s \neq 0. \tag{127}$$

The reduced matrix element $(nl\,\|\delta H^0_0\|\,nl)$ is identically zero for purely aspherical perturbations. Thus for a model containing purely aspherical perturbations equations (125)–(127) together imply that

$$\sum_{j=1}^{2l+1}\delta\omega_j = 0, \tag{128}$$

which is the desired result. Equation (128) is known as the diagonal sum rule. It implies that the average frequency of an isolated multiplet is simply the degenerate frequency of the multiplet; i.e.

$$\frac{1}{2l+1}\sum_{j=1}^{2l+1}(\omega_{nl}+\delta\omega_j) = \omega_{nl}. \tag{129}$$

It is important to note that the centripetal acceleration term in the equation of motion contains a spherically symmetric component. Therefore, this term and all other spherically symmetric terms will not satisfy the diagonal sum rule.

Second, we consider the case of cross-coupling between two or more multiplets. We assume that all SNRNMAIS modes k that compose the multiplets are members of the eigenspace K. For example, if we consider the multiplets ${}_nS_l$ and ${}_{n'}S_{l'}$, $\dim(K) = 2(l+l'+1)$. Proceeding as before, we take the trace of the supermatrix $\boldsymbol{Z}$ (equation (67)) to obtain

$$\operatorname{tr}\boldsymbol{Z} = \sum_{k\in K}\left[H^{m'm}_{n'n,l'l} - (\omega^2_{\text{ref}} - \omega^2_{nl})\right]\delta_{n'n}\,\delta_{l'l}\,\delta_{m'm} = 2\omega_{\text{ref}}\sum_{j=1}^{\dim(K)}\delta\omega_j, \tag{130}$$

where we have used equation (71) and for simplicity we have set the normalization constants N_k to unity. By virtue of equations (126) and (127), equation (130) can be rewritten

$$\sum_{k\in K}(\omega^2_{nl} - \omega^2_{\text{ref}})\,\delta_{n'n}\,\delta_{l'l}\,\delta_{m'm} = 2\omega_{\text{ref}}\sum_{j=1}^{\dim(K)}\delta\omega_j, \tag{131}$$

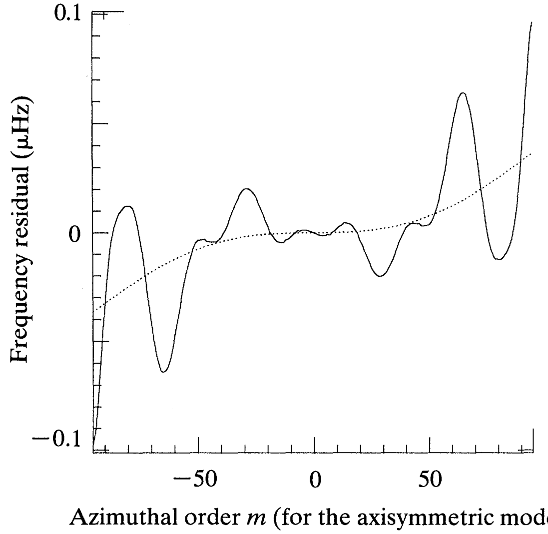

Figure 4. The solid line is the difference in eigenfrequency between an acoustic mode of Glatzmaier's (1984) model of convection and the corresponding eigenfrequency of an acoustic mode of the differentially rotating model ordered by frequency perturbation. The multiplet considered here is ${}_8S_{95}$. Frequency shifts caused by this model of convection are probably below the observable threshold. However, since Glatzmaier's model contains no flows in the top 15% of the convection zone where convective velocities are greatest, the frequency shifts shown here should be considered lower bounds. The dotted line is the residual of the frequency profiles of the two models as given by their respective Legendre polynomial fits each of which was truncated at degree $N = 5$ (frequency splitting data is often represented in terms of low-degree Legendre polynomials by observers). This figure illustrates the failure of the low-degree polynomial fit to capture the high-order behaviour of the actual frequency residuals.

so that
$$\sum_{n,l \in K} (2l+1)\,\omega_{nl}^2 = \sum_{i=1}^{\dim(K)} (2\omega_{\text{ref}}\,\delta\omega_j + \omega_{\text{ref}}^2), \tag{132}$$

where by equation (70) the terms $2\omega_{\text{ref}}\,\delta\omega_j + \omega_{\text{ref}}^2$ are simply the squared frequencies of the non-SNRNMAIS modes. Thus the average frequency of the modes is given by
$$\frac{1}{\dim(K)} \sum_{j=1}^{\dim(K)} (2\omega_{\text{ref}}\,\delta\omega_j + \omega_{\text{ref}}^2) = \frac{1}{\dim(K)} \sum_{n,l \in K} (2l+1)\,\omega_{nl}^2, \tag{133}$$

which is the generalization of the diagonal sum rule in equation (129), and is what we call the super-diagonal sum rule.

(*e*) *An example calculation*

In Lavely & Ritzwoller (1992) we present extensive numerical results using a model of convection generated by Glatzmaier (1984). Here, we simply show that the frequency splitting of a given multiplet under degenerate perturbation theory displays a symmetry property. In particular, the frequency perturbations produced by an odd-degree toroidal velocity field when ordered from the smallest to the largest perturbation had odd symmetry with respect to the degenerate frequency. This is illustrated in figure 4 for the multiplet ${}_8S_{95}$. In this calculation we used Glatzmaier's toroidal velocity field as given by the expansion coefficients w_s^t for $(1 \leqslant s \leqslant 30;$

$-s \leqslant t \leqslant s$), odd s. The largest frequency perturbation with respect to the differential rotation is approximately 0.1 μHz. This is probably an under-prediction since the velocity model contains no flows in the top 15% of the convection zone where convective velocities are greatest.

8. Differential rotation under quasi-degenerate perturbation theory

The estimation of the differential rotation profile of the Sun from measurements of p-mode frequencies has been of central importance within helioseismology (Brown *et al.* 1989; Christensen-Dalsgaard *et al.* 1990; Thompson 1990). The inversion of the frequencies for the differential rotation requires a forward model that accurately relates the data to model parameters that characterize the rotation profile. To date, all solutions of this problem have used degenerate perturbation theory (Cowling & Newing 1949; Ledoux 1951; Hansen *et al.* 1977; Gough 1982; Brown 1985; Ritzwoller & Lavely 1991). In this section, we discuss the application of quasi-degenerate perturbation theory to differential rotation. In §8*a*, we specialize the results of §§5 and 6 to obtain the general matrix element for differential rotation. In §8*b*, we present the selection rules that govern the sets of SNRNMAIS modes that will couple due to a given degree s of differential rotation. For illustrative purposes, we show in §8*c* how general matrix elements compose the supermatrix for the case in which SNRNMAIS modes from the two multiplets ${}_{n'}S_1$ and ${}_{n}S_3$ are allowed to couple; i.e. when the eigenspace for representing the perturbed eigenfunction is spanned by the eigenfunctions of only two multiplets. In §8*d*, we obtain the frequency splitting formulae of Ritzwoller & Lavely (1991) as a special case of the results in §8*a*.

(*a*) *The general matrix element for differential rotation*

The general matrix element that governs differential rotation can be obtained by restricting the velocity field $\boldsymbol{u}_0$ (defined by (23)) to include only those terms that represent the velocity field of differential rotation. This velocity field is divergence free, axisymmetric, and symmetric about the equatorial plane of the Sun. Thus, differential rotation is represented with odd-degree toroidal expansion coefficients $w_s^0(r)$:

$$\boldsymbol{u}_0(\boldsymbol{r}) = \sum_{s=1,3,5,\ldots}^{\infty} -w_s^0(r)\,\hat{\boldsymbol{r}} \times \nabla_1 Y_s^0(\theta, \phi). \tag{134}$$

Let $\tilde{C}^{m'm}_{n'n,\,l'l}$ denote the general matrix element that governs modal interactions due to the first-order effect (in $\boldsymbol{u}_0$) of differential rotation alone. It may be obtained by discarding all terms in $C^{m'm}_{n'n,\,l'l}$ (equation (D 17)) except for those containing the expansion coefficients $w_s^0(r)$ for ($s = 1, 3, 5, \ldots$):

$$\tilde{C}^{m'm}_{n'n,\,l'l} = \delta_{m'm}(-1)^{m'} 8\pi\omega_{\mathrm{ref}}\gamma_{l'}\gamma_l \sum_{s=1,3,5,\ldots}^{\infty} \gamma_s \begin{pmatrix} s & l' & l \\ 0 & m & -m \end{pmatrix} \int_0^{R_\odot} \rho_0\, w_s^0(r)\, T_s(r) r^2\, \mathrm{d}r, \tag{135}$$

where the kernel $T_s(r)$ is given by equation (98).

By using the properties of the Wigner $3j$ symbols cited in Appendix C and the definitions of the factors Ω_N^k and $B_{l'l''l}^{(N)\pm}$, we may rewrite $T_s(r)$ in the form

$$T_s(r) = -(1-(-1)^{l'+l+s})\,\Omega_0^l\,\Omega_0^{l'}\begin{pmatrix} s & l' & l \\ 0 & 1 & -l \end{pmatrix}$$
$$\times r^{-1}\{U'V+V'U-U'U-\tfrac{1}{2}V'V[l(l+1)+l'(l'+1)-s(s+1)]\}. \quad (136)$$

The Wigner $3j$ symbols in equations (135) and (136) can be computed either with the algorithm of Zare (1989) or by the following recursion relation:

$$\begin{pmatrix} s+1 & l' & l \\ 0 & -m & m \end{pmatrix} = \frac{A_{ll's}}{\{[(l+l'+1)^2-(s+1)^2][(s+1)^2-(l-l')^2]\}^{\frac{1}{2}}}, \quad (137)$$

where

$$A_{ll's} = -(4s+2)\,m\begin{pmatrix} s & l' & l \\ 0 & -m & m \end{pmatrix} - \{[(l+l'+1)^2-s^2][s^2-(l-l')^2]\}^{\frac{1}{2}}\begin{pmatrix} s-1 & l' & l \\ 0 & -m & m \end{pmatrix}, \quad (138)$$

which can be derived from equation ($5a$) of Schulten & Gordon (1975).

(b) *Selection rules and the supermatrix*

Since s is always odd for differential rotation, it follows from selection rule 3 that coupling between modes with harmonic degrees l and l' through degree s of differential rotation can occur only if $l' = l \pm 2j$, where j is an integer. Upper and lower bounds on j follow from the triangle inequalities in selection rule 1:

$$l' = l \pm 2j \quad \text{where} \quad (0 \leqslant j \leqslant \tfrac{1}{2}(s-1)). \quad (139)$$

Thus, for $s=1$, coupling is restricted to $l'=l$; for $s=3$, coupling is restricted to $l'=l$, $l\pm 2$; for $s=5$, coupling is restricted to ($l'=l$, $l\pm 2$, $l'=l\pm 4$); and so on for higher degree s. In addition, selection rule 2 assures us that $m'=m$ since $t=0$ for differential rotation. There are no selection rules on the radial orders n and n'. The dependence of $\tilde{C}_{n'n,l'l}$ on n and n' is given entirely by the scalar eigenfunctions in the kernel $T_s(r)$. However, the quasi-degeneracy condition in equation (52) assures us that only nearly degenerate modes can couple strongly.

(c) *An example of quasi-degenerate coupling*

To illustrate the structure of the supermatrix $\boldsymbol{Z}$ (equation (67)) and the coupling that is allowed by the selection rules for a particularly simple case of quasi-degenerate coupling, we consider the coupling of all modes of the SNRNMAIS solar model that compose the multiplets ${}_nS_1$ and ${}_{n'}S_3$ (where n and n' shall remain unspecified). Thus we define the eigenspace K to be composed of the modes ($n, l = 1$, $-1 \leqslant m \leqslant 1$) and ($n', l' = 3$, $-3 \leqslant m' \leqslant 3$). There are two types of coupling that can occur; self-coupling ($n'=n$, $l'=l$) and cross-coupling ($n' \neq n$, or $l' \neq l$). Since $m'=m$ by selection rule 2, the supermatrix will display a banded structure and modes from the multiplet ${}_{n'}S_3$ with $|m'|>1$ will not couple with modes from the multiplet ${}_nS_1$. From selection rule 1_{sc}, self-coupling can occur for ${}_nS_1$ only when $s=1$; for ${}_{n'}S_3$ when $s=1, 3, 5$. Cross-coupling can occur only for $s=3$. The two cross-coupling pairs are given by ([$l=1$, $m=1$], [$l'=3$, $m'=1$]), and ([$l=1$, $m=-1$], [$l'=3$, $m'=-1$]). The modes with $m=m'=0$ do not couple since in this case their general matrix elements identically vanish due to the selection rule in equation (C 42).

Before constructing the supermatrix for the special case described above, we introduce new notation for clarity. For the remainder of this subsection, we suppress the subscripts n and n' in the matrix element $\tilde{C}^{m'm}_{n'n,\,l'l}$ and rewrite it as $\tilde{C}^{m'm}_{(l',l)}$. We define the squared reference frequency ω^2_{ref} as the average of the squares of the degenerate frequencies of the two multiplets:

$$\omega^2_{\text{ref}} = \tfrac{1}{2}[\omega^2_{n1} + \omega^2_{n'3}]. \tag{140}$$

Defining the frequency difference factors as follows:

$$-\Delta = 2(\omega^2_{n1} - \omega^2_{\text{ref}}), \tag{141}$$

$$\Delta = 2(\omega^2_{n'3} - \omega^2_{\text{ref}}), \tag{142}$$

the supermatrix $\boldsymbol{Z}$ (equation (67)) in the context of the above discussion (neglecting the Coriolis force, centripetal acceleration, etc.) can be written

$$\left[\begin{array}{ccc|ccccccc}
\tilde{C}^{-1,-1}_{(1,1)} - \frac{1}{2}\Delta & 0 & 0 & 0 & 0 & \tilde{C}^{-1,-1}_{(1,3)} & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2}\Delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \tilde{C}^{1,1}_{(1,1)} - \frac{1}{2}\Delta & 0 & 0 & 0 & 0 & \tilde{C}^{1,1}_{(1,3)} & 0 & 0 \\
\hline
0 & 0 & 0 & \tilde{C}^{-3,-3}_{(3,3)} + \frac{1}{2}\Delta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \tilde{C}^{-2,-2}_{(3,3)} + \frac{1}{2}\Delta & 0 & 0 & 0 & 0 & 0 \\
\tilde{C}^{-1,-1}_{(1,3)} & 0 & 0 & 0 & 0 & \tilde{C}^{-1,-1}_{(3,3)} + \frac{1}{2}\Delta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\Delta & 0 & 0 & 0 \\
0 & 0 & \tilde{C}^{1,1}_{(1,3)} & 0 & 0 & 0 & 0 & \tilde{C}^{1,1}_{(3,3)} + \frac{1}{2}\Delta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{C}^{2,2}_{(3,3)} + \frac{1}{2}\Delta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{C}^{3,3}_{(3,3)} + \frac{1}{2}\Delta
\end{array}\right]$$

where we have set the normalization constants N_k to unity. The vertical and horizontal lines partition the supermatrix into four blocks; the upper left-hand block is the self-coupling block for the multiplet ${}_nS_1$, the lower right-hand block is the self-coupling block for the multiplet ${}_{n'}S_3$, and the off-diagonal blocks are the quasi-degenerate coupling blocks. Due to the selection rules, most of the components of these blocks are identically zero. Each off-diagonal term links the opposing diagonal elements. Thus, the general matrix element $\tilde{C}^{1,1}_{(1,3)}$ signifies the coupling of the ($l = 1$, $m = 1$) and ($l = 3$, $m = 1$) modes. By equation (117), the coupling strength coefficient ν in the case of the mode pair ($l = 1$, $m = 1$) and ($l = 3$, $m = 1$) is given by

$$\nu \approx (\tilde{C}^{1,1}_{(1,3)})^2/(\tilde{C}^{1,1}_{(1,1)} - \tilde{C}^{1,1}_{(3,3)} + \Delta). \tag{143}$$

The numerical application of this formalism to determine quantitatively the size of the coupling terms will be discussed in a later paper. A qualitative discussion, however, is sufficient to establish that quasi-degenerate coupling through differential rotation will have little effect on modal frequencies and can be ignored for all practical purposes. An inspection of the solar dispersion diagram reveals that for intermediate l modes ($l \approx 50$), $\Delta \approx 30$ μHz for $l' = l+2$. Since the coupling terms will be no larger than *ca.* 0.1 μHz, the eigenfrequency perturbation caused by quasi-degenerate coupling will be less than $\nu \approx 1$ nHz. This will be a slight frequency repulsion of every other mode within the multiplets with harmonic degrees l and $l+2$. That is, the frequency of each affected l mode will decrease and the frequency of each affected $l+2$ mode will increase. However, the interaction of each l mode with each $l-2$ mode of the same m, will act on the l modes with the opposite sign and will counteract this already small effect. Thus almost certainly, the seismic effect of low-degree differential rotation through quasi-degenerate coupling can be practically neglected. This is not true for a general non-axisymmetric flow field, since there will be many possible near degeneracies (Lavely & Ritzwoller 1992).

(*d*) *Differential rotation under degenerate perturbation theory*

The general matrix element governing the splitting caused by differential rotation under degenerate perturbation theory is a special case of equation (135). Ritzwoller & Lavely (1991) treated this problem in detail. Their formulae can be recovered by (1) setting $\omega_{\text{ref}} = \omega_{nl}, n' = n,\ l' = l$, and $m' = m$ in equation (65), (2) defining the eigenspace K to include only the $(2l+1)$ modes $(n, l, -l \leqslant m \leqslant l)$, (3) defining the splitting matrix $\boldsymbol{H}_{nn,ll}$ (equation (79)) to include contributions from the splitting matrices $\boldsymbol{B}_{nn,ll}$ (equation (83)) and $\boldsymbol{C}_{nn,ll}$ (equation (84)) only, and (4) including only the axisymmetric, odd-degree, toroidal component of the velocity field $\boldsymbol{u}_0$ that appears in the splitting matrix $\boldsymbol{C}_{nn,ll}$. Performing operations (1)–(4), the general matrix elements for differential rotation and the Coriolis force under degenerate perturbation theory are given by

$$\tilde{C}^{m'm}_{nn,ll} = \delta_{m'm}(-1)^{m'} 8\pi\omega_{nl}\gamma_l^2 \sum_{s=1,3,5,\ldots}^{\infty} \gamma_s \begin{pmatrix} s & l & l \\ 0 & m & -m \end{pmatrix} \int_0^{R_\odot} \rho_0\, w_s^0(r)\, T_s(r)\, r^2\, \mathrm{d}r, \tag{144}$$

$$B^{m'm}_{nn,ll} = \delta_{m'm}\, 2\omega_{nl}\, m\Omega \int_0^{R\odot} \rho_0 (2UV+V^2)\, r^2\, \mathrm{d}r, \tag{145}$$

and $T_s(r)$ is given by

$$T_s(r) = -(1-(-1)^s)(\Omega_0^l)^2 \begin{pmatrix} s & l & l \\ 0 & 1 & -1 \end{pmatrix} r^{-1}[2UV - U^2 - \tfrac{1}{2}V^2[2l(l+1) - s(s+1)]. \tag{146}$$

The supermatrix eigenvalue problem in equation (65) can be written:

$$\sum_{m=-l}^{l} a_m\, [B^{m'm}_{nn,ll} + \tilde{C}^{m'm}_{nn,ll}] = 2N_k\, \omega_{nl} \sum_{m=-l}^{l} a_m\, \delta\omega\, \delta_{m'm}. \tag{147}$$

For reasonably small s, analytic expressions for Wigner $3j$ symbols with the special forms in equations (144) and (146) can be obtained by specializing the recursion relationship in equation (137). The latter can be simplified considerably by setting

$$H_s^m F_s = (-1)^{(l-m)} \begin{pmatrix} s & l & l \\ 0 & m & -m \end{pmatrix}, \tag{148}$$

where we have defined

$$F_s = \left[\frac{(2l-s)!}{(2l+s+1)!}\right]^{\frac{1}{2}}. \tag{149}$$

Substituting equation (148) into equation (137) and setting $l' = l$, a recursion relationship for the H_s^m coefficients can easily be derived:

$$H^m_{s+1} = (s+1)^{-1}[2(2s+1)\, mH_s^m - s(4l(l+1)+1-s^2)\, H^m_{s-1}]. \tag{150}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

To initiate the recursion we require the values of H_0^m and H_1^m. These are given by ($H_0^m = 1$) and by ($H_1^m = 2m$). Values of H_s^m for ($0 \leqslant s \leqslant 11$) have been tabulated in Ritzwoller & Lavely (1991) (see (39)–(44) and (A 4)–(A 9) in that paper).

In this notation, $C_{nn,ll}^{m'm}$ can be rewritten

$$\tilde{C}_{nn,ll}^{m'm} = \delta_{m'm}\, 4(2l+1)\, \omega_{nl} (\Omega_0^l)^2 \sum_{s=1,3,5,\ldots}^{\infty} \gamma_s H_s^1 H_s^m F_s^2 \times \int_0^{R_\odot} \rho_0\, w_s^0(r)\, [2UV - U^2 - V^2[l(l+1) - \tfrac{1}{2}s(s+1)]]\, r\, \mathrm{d}r, \quad (151)$$

and we have used the fact that s is an odd integer. The solution to the forward problem given in Ritzwoller & Lavely (1991) (their eq. (32)) is equivalent to the relation

$$\omega_{nl}^m = \omega_{nl} + (1/2N_k\, \omega_{nl})\, [\tilde{C}_{nn,ll}^{mm} + B_{nn,ll}^{mm}] - m\Omega, \quad (152)$$

where $\tilde{C}_{nn,ll}^{m'm}$ accounts for the effect of differential rotation, $B_{nn,ll}^{m'm}$ accounts for the Coriolis force, and the term $m\Omega$ transforms the frequency from the corotating to an inertial frame in accordance with equation (20).

We note that the w_1^0 expansion coefficient in Ritzwoller & Lavely (1991) is defined to include the velocity field due to rigid rotation whereas the w_1^0 expansion coefficient in this paper is defined to include only departures from rigid rotation in the radial coordinate. We also address a potential point of confusion. Some authors define the time dependence of a mode to be given by $\exp(-\mathrm{i}\omega t)$ (e.g. Gough 1982), whereas others define the time dependence to be given by $\exp(\mathrm{i}\omega t)$ (e.g. Hansen *et al.* 1977). These differing conventions lead to differing signs on the right-hand side of equation (152). The convention used in this paper and in Ritzwoller & Lavely (1991) is consistent with that of Hansen *et al.* (1977).

9. Theoretical wavefields

In this section we show how to construct theoretical wavefields for SNRNMAIS and non-SNRNMAIS solar models. The calculation requires a representation of the source of acoustic energy. We adopt the prevailing view that the solar p-modes are excited by acoustic noise generated by turbulent convection near the solar surface. Goldreich & Kumar (1988) have shown that acoustic emissions vary roughly as the eighth power of the Mach number. Consequently, Brown (1990) has suggested that most of the emission originates from that small fraction of the flow volume containing the very highest velocity flows. By assuming a simple probability distribution for the flow speeds, Brown (1990) estimates that this fraction is approximately 0.5%. Brown (1990) also suggests that the sustained flows that produce acoustic energy have lifetimes that are a fraction of the solar granulation turnover time. It follows that the acoustic source is highly localized in space and short-lived in time, rather than smooth, and temporally continuous. We adopt this argument to obtain an expression for the acoustic displacement field in which the acoustic energy is generated by a superposition of discrete sources each of which are localized in space and time. In §9*a* we derive an expression for a theoretical wavefield of a SNRNMAIS solar model and generalize the result to a non-SNRNMAIS solar model in §9*b*.

(a) Theoretical wavefield for a SNRNMAIS solar model

From Biot (1965) and Dziewonski & Woodhouse (1983), it can be shown that in a perfect fluid the most general local linear relationship between stress and strain is given by

$$T_{ij}(\boldsymbol{u}) = \delta_{ij}[\kappa_0(r) - P_0(r)]\,\partial_i u_i + P_0(r)\,\partial_i u_j, \tag{153}$$

where T_{ij} is the stress tensor and it is assumed that the medium is perfectly spherically symmetric and, in its equilibrium configuration, the stress is purely hydrostatic. The quantity u_i is the ith component of the displacement. Equation (153) will break down when nonlinear processes are important. Near the solar surface where acoustic energy generation is most significant, the Mach number approaches its peak value of approximately 0.3. The turbulent flows in these regions will lead to strains that are not linearly related to the corresponding stresses so that equation (153) would be invalid. Similarly, the linear stress–strain relationship breaks down in the fault ruptures that generate terrestrial seismic waves. For this reason, in the modelling of the terrestrial source process, Backus & Mulcahy (1976*a*, *b*) introduced the stress glut tensor Γ_{ij} defined by the relation

$$\Gamma_{ij} = T_{ij}(\boldsymbol{s}) - \tilde{T}_{ij}. \tag{154}$$

The symmetric tensor $\boldsymbol{\Gamma}$ represents the difference between the stress predicted by equation (153) when the true physical displacement $\boldsymbol{s}$ is substituted, and the actual stress $\tilde{\boldsymbol{T}}$. In the picture of Goldreich & Kumar (1988) and Brown (1990), the stress glut tensor vanishes in all regions of the Sun except those where the turbulence is most vigorous. The source regions are defined by those regions of space where $\boldsymbol{\Gamma}(\boldsymbol{r}, t)$ is non-zero. Clearly, $\boldsymbol{\Gamma}(\boldsymbol{r}, t)$ will display a very complicated dependence upon position and time.

The force density that drives the acoustic oscillations is simply given by the divergence of the stress glut tensor; i.e.

$$\boldsymbol{f}(\boldsymbol{r}, t) = -\nabla \cdot \boldsymbol{\Gamma}(\boldsymbol{r}, t). \tag{155}$$

Thus the equation of motion governing the forced oscillations of the Sun is given by

$$\rho_0 \ddot{\boldsymbol{u}} = \mathscr{L}_0 \boldsymbol{u} - \nabla \cdot \boldsymbol{\Gamma}, \tag{156}$$

where $\mathscr{L}_0$ is defined by equation (B 20).

When $\boldsymbol{f} = \boldsymbol{0}$, the solution to equation (156) is given by $\boldsymbol{s}_k(\boldsymbol{r}) \exp(\mathrm{i}\omega_k t)$. These modes are simply the free oscillations of the SNRNMAIS solar model. To solve the inhomogeneous equation (156), we introduce directly into the equation of motion the effect of intrinsic attenuation and rewrite equation (156) as:

$$\rho_0 \ddot{\boldsymbol{u}} + 2\alpha\rho_0 \dot{\boldsymbol{u}} - \mathscr{L}_0 \boldsymbol{u} = \boldsymbol{f}, \tag{157}$$

where the term in $\dot{\boldsymbol{u}}$ models attenuation. Following Dziewonski & Woodhouse (1983), we seek $\boldsymbol{u}(\boldsymbol{r}, t)$ in terms of an eigenfunction expansion:

$$\boldsymbol{u}(\boldsymbol{r}, t) = \sum_k a_k(t)\, \boldsymbol{s}_k(\boldsymbol{r}), \tag{158}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

and solve for $a_k(t)$. Upon inserting equation (158) into equation (157), taking the inner product of the resulting expression with $\boldsymbol{s}_k^*(\boldsymbol{r})$, and using the orthogonality of the SNRNMAIS modes (equation (4)), we find

$$\ddot{a}_k(t)+2\alpha_k\,\dot{a}_k(t)+\omega_k^2\,a_k(t)=F_k(t)/N_k, \tag{159}$$

where the source function $F_k(t)$ is given by

$$F_k(t)=\int \boldsymbol{s}_k^*(\boldsymbol{r})\cdot\boldsymbol{f}(\boldsymbol{r},t)\,\mathrm{d}^3\boldsymbol{r}, \tag{160}$$

where N_k is defined by equation (5) and we have used equation (51). The attenuation coefficient α_k is related to the quality factor Q_k by the expression $\alpha_k=\omega_k/2Q_k$.

Before proceeding further, it is useful to consider in greater detail the nature of the source process. Brown (1990) suggests that a typical linear dimension of the source is given by a fraction of the dimension of a typical solar granule which is approximately 1200 km. In addition, the source lifetime is given by a fraction of a solar granule turnover time which is approximately 480 s. A typical wavelength of acoustic modes for $l=100$ is approximately 44000 km. In addition, the typical modal period is 300 s. Thus, for low and intermediate degree modes, the linear dimensions of the source region are smaller than the smallest modal wave-lengths included in equation (158). However, the source duration time is likely to be comparable with or greater than the modal periods of interest. Thus we are justified in representing the source volume as a spatial delta-function, but we will retain the arbitrariness of the source-time function. The time history of individual source volumes will be assumed to be finite in duration.

We define an acoustic source as any deformation process that violates the linear stress–strain relationship in equation (153). The acoustic wavefield is generated by multiple sources so that the total body force density is given by the sum of force densities for individual sources:

$$\boldsymbol{f}(\boldsymbol{r},t)=\sum_{\sigma}\boldsymbol{f}_{\sigma}(\boldsymbol{r},t), \tag{161}$$

where σ is the source index. For multiple sources the source function $F_k(t)$ in equation (160) becomes

$$F_k(t)=\sum_{\sigma}F_k^{\sigma}(t), \tag{162}$$

where

$$F_k^{\sigma}(t)=\int \boldsymbol{s}_k^*(\boldsymbol{r})\cdot\boldsymbol{f}_{\sigma}(\boldsymbol{r},t)\,\mathrm{d}^3\boldsymbol{r}. \tag{163}$$

By the arguments above, each source function $F_k^{\sigma}(t)$ has finite time extent. Thus,

$$F_k^{\sigma}(t)\neq 0 \quad \text{only in the time interval} \quad t_1^{\sigma}\leqslant t\leqslant t_2^{\sigma}. \tag{164}$$

Equation (159) represents an inhomogeneous damped harmonic oscillator equation with constant coefficients. Methods of solution for this type of equation are well known. In terms of the Green's function method, the solution is given by

$$a_k(t)=\frac{1}{N_k}\int_{-\infty}^{t}G_k(t,t')\,F_k(t')\,\mathrm{d}t' \tag{165}$$

$$=\frac{1}{N_k}\sum_{\sigma}\int_{-\infty}^{t}G_k(t,t')\,F_k^{\sigma}(t')\,\mathrm{d}t', \tag{166}$$

where the causal Green's function, $G_k(t,t')$, for a damped harmonic oscillator (Kumar *et al.* 1988) is given by

$$G_k(t,t') = \tilde{\omega}_k^{-1} H(t-t') \exp[-\alpha_k(t-t')] \sin \tilde{\omega}_k(t-t'), \tag{167}$$

where $H(t)$ is the Heaviside step function, and

$$\tilde{\omega}_k^2 = \omega_k^2 - \alpha_k^2. \tag{168}$$

For typical solar p modes, $\omega_k \gg \alpha_k$, and therefore, we replace $\tilde{\omega}_k$ with ω_k so that $G_k(t,t')$ becomes

$$G_k(t,t') = \omega_k^{-1} H(t-t') \exp[-\alpha_k(t-t')] \sin \omega_k(t-t'). \tag{169}$$

Integrating equation (166) by parts, and assuming that the time derivative of each source function $F_k^\sigma(t)$ vanishes before the source began to be active and also after it has ceased to act (these time intervals are specified in (164)), we obtain

$$a_k(t) = \sum_\sigma \int_{-\infty}^{\infty} h_k(t-t') \dot{F}_k^\sigma(t')\, \mathrm{d}t' \tag{170}$$

$$= h_k(t) \bigstar \sum_\sigma \dot{F}_k^\sigma(t), \tag{171}$$

where $\bigstar$ denotes the convolution operator, the overdot signifies a time derivative here, and

$$h_k(t) = (1 - \mathrm{e}^{-\alpha_k t} \cos(\omega_k t))/N_k \omega_k^2. \tag{172}$$

The expression for $a_k(t)$ in equation (171) represents the time history of the acoustic displacement-field for a superposition of source processes and is seen to be a simple convolution of a harmonic resonance with the sum of all source functions. In terms of the stress glut rate tensor, $\dot{F}_k^\sigma(t)$ is given by

$$\dot{F}_k^\sigma(t) = -\int \boldsymbol{s}^{*(k)}(\boldsymbol{r}) \cdot [\nabla \cdot \dot{\boldsymbol{\Gamma}}^\sigma(\boldsymbol{r},t)]\, \mathrm{d}^3\boldsymbol{r}. \tag{173}$$

By combining equations (170), and (173), $a_k(t)$ is expressed as

$$a_k(t) = -h_k(t) * \sum_\sigma \int_{V_\sigma} \boldsymbol{s}^{*(k)}(\boldsymbol{r}) \cdot [\nabla \cdot \dot{\boldsymbol{\Gamma}}^\sigma(\boldsymbol{r},t)]\, \mathrm{d}^3\boldsymbol{r}, \tag{174}$$

where it has been assumed that $\dot{\boldsymbol{\Gamma}}^\sigma(\boldsymbol{r},t)$ vanishes outside of the volume region V_σ. By applying the divergence theorem, $a_k(t)$ can be rewritten

$$a_k(t) = h_k(t) * \sum_\sigma \int_{V_\sigma} \dot{\boldsymbol{\Gamma}}^\sigma(\boldsymbol{r},t) : \boldsymbol{e}^{(k)*}(\boldsymbol{r})\, \mathrm{d}^3\boldsymbol{r}, \tag{175}$$

where $\boldsymbol{e}^{(k)}(\boldsymbol{r})$ is the strain tensor and is defined by the relation

$$\boldsymbol{e}^{(k)}(\boldsymbol{r}) = \tfrac{1}{2}[\nabla \boldsymbol{s}^{(k)}(\boldsymbol{r}) + \boldsymbol{s}^{(k)}(\boldsymbol{r})\nabla]. \tag{176}$$

As argued previously, the volume region V_σ is small relative to a typical modal wavelength appearing in equation (158). For example, the dimension of the source region relative to the wavelength of an $l = 100$ mode is approximately 2%. The relative smallness of the spatial region suggests the substitution

$$\dot{\boldsymbol{\Gamma}}^\sigma(\boldsymbol{r},t) \rightarrow \dot{\boldsymbol{\Gamma}}^\sigma(\boldsymbol{r},t)\, \delta^3(\boldsymbol{r}-\boldsymbol{r}_\sigma) \tag{177}$$

in equation (175) to obtain

$$a_k(t) = h_k(t) * \sum_\sigma \dot{\boldsymbol{\Gamma}}^\sigma(\boldsymbol{r}_\sigma, t) : \boldsymbol{e}^{(k)*}(\boldsymbol{r}_\sigma). \tag{178}$$

Clearly, $a_k(t)$ is the convolution of the harmonic resonance of the mode with the source-time history of the stress glut rate tensor projected onto the appropriate strain components. Equation (178) together with equation (158) define the total displacement field for the SNRNMAIS solar model:

$$\boldsymbol{u}(\boldsymbol{r}, t) = \sum_k h_k(t) * \sum_\sigma \dot{\boldsymbol{\Gamma}}^\sigma(\boldsymbol{r}_\sigma, t) : \boldsymbol{e}^{(k)*}(\boldsymbol{r}_\sigma)\, \boldsymbol{s}_k(\boldsymbol{r})$$

$$= \sum_k [\cos(\omega_k t)\, \mathrm{e}^{-\alpha_k t}] * \sum_\sigma a_k^\sigma(\boldsymbol{r}_\sigma, t)\, \boldsymbol{s}_k(\boldsymbol{r}), \tag{179}$$

where the source coefficient a_k^σ is defined by the relation

$$a_k^\sigma(\boldsymbol{r}_\sigma, t) = -\boldsymbol{M}^\sigma(\boldsymbol{r}_\sigma, t) : \boldsymbol{e}^{(k)*}(\boldsymbol{r}_\sigma)/N_k\, \omega_k^2, \tag{180}$$

where $\boldsymbol{M}^\sigma(\boldsymbol{r}_\sigma, t) = \dot{\boldsymbol{\Gamma}}^\sigma(\boldsymbol{r}_\sigma, t)$ is the moment rate tensor of the σth source, and where we have substituted from equation (172) retaining only the non-static term.

(b) *Theoretical wavefield for a non-SNRNMAIS solar model under degenerate perturbation theory*

Following Woodhouse & Girnius (1982), we now show how equation (179) may be transformed to yield the expression for a wavefield in a non-SNRNMAIS solar model. For clarity of presentation, we consider the special case of self-coupling (degenerate perturbation theory) in which only those modes that compose the multiplet ${}_nS_l$ are allowed to couple. The procedure consists of rotating the vector of source coefficients with elements given by a_k^σ, $k = (n, l, -l \leqslant m \leqslant l)$, and the SNRNMAIS basis functions $\boldsymbol{s}_{nl}^m(\boldsymbol{r})$ into the normal coordinates of the non-SNRNMAIS solar model, and by introducing the split frequency spectrum. If we normalize the splitting matrix for the multiplet ${}_nS_l$ by the factor $2\omega_{nl}$, then by equation (124), the diagonal matrix $\boldsymbol{\varLambda}$ containing the frequencies $\delta\omega_j$ associated with the jth eigenvector of the splitting matrix is given by

$$\boldsymbol{\varLambda} = \boldsymbol{A}^\dagger \boldsymbol{H}_{nn,ll} \boldsymbol{A}, \tag{181}$$

where $\boldsymbol{A}$ is the eigenvector matrix that diagonalizes the normalized splitting matrix $\boldsymbol{H}_{nn,ll}$ and where the dagger symbol (†) denotes the complex conjugate transpose. From Woodhouse & Girnius (1982), the displacement field generated by the σth source, and associated with the multiplet ${}_nS_l$ is given by

$$\boldsymbol{u}_{nl}^\sigma(\boldsymbol{r}, t) = (\boldsymbol{s}_{nl}^{\mathrm{T}} \boldsymbol{A}) \exp[\mathrm{i}(\boldsymbol{\varLambda} + (\omega_{nl} + \mathrm{i}\alpha_{nl})\, \boldsymbol{I})\, t] * (\boldsymbol{A}^\dagger \boldsymbol{a}^\sigma(t)), \tag{182}$$

where T and $\boldsymbol{I}$ denote, respectively, the transpose operation and the identity matrix, $\boldsymbol{a}^\sigma$ and $\boldsymbol{s}_{nl}$ denote, respectively, the vector of time-dependent source coefficients and of basis functions for $(-l \leqslant m \leqslant l)$, and where we have assumed $\alpha_k = \alpha_{nl}$.

As discussed in §2, the transformation to the stationary frame is accomplished by the variable transformation $\phi \rightarrow \phi - \varOmega t$ where $\varOmega$ is the average angular rotation rate at the solar surface. Noting that the azimuthal (ϕ) dependence of $\boldsymbol{s}_{nl}^m(\boldsymbol{r})$ is given by $\exp(\mathrm{i}m\phi)$, and using index notation for clarity, we obtain

$$\boldsymbol{u}_{nl}^{\sigma}(\boldsymbol{r},t) = \sum_{j=1}^{2l+1} \sum_{m=-l}^{l} A_{mj}\, \boldsymbol{s}_{nl}^m(\boldsymbol{r}) \sum_{i=1}^{2l+1} A_{ji}^{\dagger}\, a_i^{\sigma}(t) * \lambda_j^m(t), \tag{183}$$

where the lorentzian function $\lambda_j^m(t)$ is given by

$$\lambda_j^m(t) = \exp\left[\mathrm{i}(\omega_{nl}+\delta\omega_j - m\Omega)\,t\right] \exp(-\alpha_{nl}\,t). \tag{184}$$

The total displacement field is

$$\boldsymbol{u}(\boldsymbol{r},t) = \sum_{\sigma} \sum_{n,l} \boldsymbol{u}_{nl}^{\sigma}(\boldsymbol{r},t), \tag{185}$$

where the sums are taken over all sources and all multiplets. The generalization of equation (185) to quasi-degenerate perturbation theory is straightforward; it suffices to increase the dimension of the eigenspace in equation (158) so that SNRNMAIS modes from more than one multiplet are included.

10. Summary and conclusions

The purpose of this paper has been to derive a theory that governs the effect of steady-state convection and associated asphericities in the elastic-gravitational variables (adiabatic bulk modulus κ, density ρ, and gravitational potential ϕ) on acoustic frequencies and displacement patterns and to present a formalism with which this theory can be applied computationally. The theory is not intended to predict modal amplitudes since these are governed, in part, by the exchange of energy between convection and acoustic waves, which is excluded since our theory is linear and since the convective flow is defined to be anelastic. We have made no simplifying assumptions about the geometric structure of the convective flow and structural asphericities, and have represented these vector and scalar fields with general global basis functions, vector and scalar spherical harmonics, respectively. We also represent the eigenfunctions of the spherical reference model (the SNRNMAIS solar model) with vector spherical harmonics. These representations allow us to use quasi-degenerate perturbation theory in a straightforward manner to derive the general matrix elements $H_{n'n,l'l}^{m'm}$ that govern the modal coupling induced by the perturbations. Formulae for the general matrix elements are presented explicitly in terms of the scalar eigenfunctions of the SNRNMAIS solar model. Thus, the use of this theory requires only the following quantities: (1) a SNRNMAIS solar model ($\kappa(r)$ and $\rho(r)$), (2) the seismic scalar eigenfunctions of the SNRNMAIS solar model (${}_nU_l(r)$, ${}_n\dot{U}_l(r)$, ${}_nV_l(r)$, ${}_n\dot{V}_l(r)$, ${}_n\delta\phi_l(r)$ and ${}_n\delta\dot{\phi}_l(r)$), and (3) the spherical harmonic representation of convection ($u_s^t(r), v_s^t(r)$, and $w_s^t(r)$) and/or asphericities in the elastic-gravitational variables ($\delta\kappa_s^t(r)$ and $\delta\rho_s^t(r)$) at each radial knot of the SNRNMAIS solar model. The general matrix elements compose the hermitian supermatrix $\boldsymbol{Z}$, whose eigenvalues are the eigenfrequency perturbations of the general non-SNRNMAIS solar model and whose eigenvector components are the expansion coefficients in the linear combination forming the eigenfunctions (or displacement patterns) in which SNRNMAIS eigenfunctions are basis functions.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The major constraint on the application of the theory presented here is that we have assumed that the convective flows and asphericities are stationary in time. Consequently, we view this paper as the first step toward a more general theory governing time-varying flows. Nevertheless, a number of the consequences of the theory will hold for time-varying flows as well. Most importantly, the selection rules listed in §7 will hold for non-stationary flows. For example, under self-coupling (or within degenerate perturbation theory), by selection rules 1_{sc} and 3_{sc}, only odd-degree s toroidal flows and even degree structural asphericities with $s \leqslant 2l$ will couple and/or split modes with harmonic degree l.

The next stage of this research will be the numerical implementation of the theory by using a realistic model of global-scale convective flow and associated structural asphericities. The goals of the work include: (1) The determination of the accuracy of degenerate perturbation theory relative to quasi-degenerate perturbation theory. (2) Numerical estimates of the effect of convective flow on helioseismic observables such as frequencies and line-widths. A preliminary result is shown in figure 4 and further calculations are presented in Lavely & Ritzwoller (1992). (3) A determination of the characteristic signatures of convection to aid observers in establishing the existence of giant cell convection. (4) An estimation of the effect of a time-varying stress glut rate on theoretical power spectra. (5) A determination of the relative importance of perturbations to the elastic-gravitational variables as compared to convective flow fields. For example, Kuhn *et al.* (1988) observed a surface temperature variation of several degrees from the solar south pole to the solar equator and hypothesized that this or a similar structure may be responsible for the non-zero even-degree frequency splitting coefficients. Given an equation of state, these temperature variations could be expressed in terms of the perturbations $\delta\rho_s^0(r)$ and $\delta\kappa_s^0(r)$. Although the depth extent of the observed temperature variation is unknown, different hypothesized depth structures could be constructed. By using the theory presented here, the general matrix elements and, hence, the splitting caused by each temperature model can be computed.

In closing, since this paper is long, it is worthwhile to present a road map through the major results. Modal notation and terminology are discussed in §1*a* and model notation and terminology are presented in §4*b*. The major assumptions of the theory are presented and discussed in §1*b*, and are justified in §§1*c* and 1*e*. The mathematical representation of convection is in equations (23)–(25) and the representation of the elastic-gravitational variables is in equations (29)–(31) and (35)–(37). The equation of motion for the perturbed model with first-order perturbations including rotation, ellipticity in the structural variables, centripetal force, convective flow, and asphericities in the elastic-gravitational variables is given by equation (50). The general forms of the general matrix element and the supermatrix are shown in equations (66) and (67), respectively, and the general matrix element for the perturbations listed in the previous sentence is in equation (76). The explicit form of the general matrix elements suitable for computation, written in terms of the scalar eigenfunctions of the SNRNMAIS solar model, can be found in equation (90) with notation and the integral kernels defined in equations (91)–(110). We consider this the main result of this paper. Three selection rules governing coupling are listed in equations (118), (120), and (121), with the self-coupling form of the selection rules in equations (119), (120), and (122). The diagonal sum rule and the super-diagonal sum rule are stated and proved in §7*d*. The general matrix element and selection rules for

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

differential rotation are in equations (135) and (139), respectively. In §9 we show how theoretical seismograms for SNRNMAIS (equation (179)) and non-SNRNMAIS (equation (183)) solar models may be computed for a model source process. All results of the paper are presented in a frame corotating with the Sun. Equation (18) can be used to construct the perturbed eigenfrequencies and eigenfunctions of a non-SNRNMAIS solar model in an inertial frame. An example of the application of equation (18) is given in equation (183) where we obtain an expression for a theoretical seismogram of a non-SNRNMAIS model in the inertial frame.

We are grateful to Addrian Van Ballegooijen for insightful criticism and numerous discussions, and for providing us with a stellar evolution code. We appreciate John Woodhouse's assistance in the derivation in Appendix D, and for adapting his normal mode program to calculate solar eigenfunctions. In addition, we wish to thank Timothy Brown, Douglas Gough, Thomas Jordan, Robert Noyes, Juri Toomre and John Woodhouse for continuous support and encouragement. We thank Edmund Bertschinger and Pawan Kumar for valuable conversations. We have adopted a suggestion made by John Wahr which simplified the proof of Rayleigh's principle in §5. The computer support provided to us by Jeremy Bloxham and Richard O'Connell is greatly appreciated. We thank Philipp Podsiadlowski for providing his solar model and Gary Glatzmaier for providing the numerical simulation of convection used to generate figure 4. Part of this work was completed at the Institute for Theoretical Physics at the University of California, Santa Barbara. This research was supported by the National Science Foundation under Grants no. PHY89-04035, no. ATM-88-05194, no. ATM-88-05546, and by NASA under Grant no. NAGW-1677. E. M. L. was partly supported by a NASA Graduate Student Fellowship. The National Center for Atmospheric Research is sponsored by the National Science Foundation.

Appendix A. The two-point boundary value problem governing the eigenfunctions and eigenfrequencies of the SNRNMAIS solar model

For clarity and uniformity of presentation, we present below the four coupled first-order differential equations that govern the eigenfunctions and eigenfrequencies of the modes of the SNRNMAIS solar model. Of course, equivalent systems of equations have been derived elsewhere; e.g. eqs (17.14)–(17.17) in Unno *et al.* (1979).

The equation of motion for the SNRNMAIS model (equation (51)) with the operator $\mathscr{L}_0$ defined by equation (B 20) together with the perturbed Poisson equation (equation (B 11)) and the perturbed continuity equation provide a coupled system of first-order ordinary differential equations that can be solved for the scalar eigenfunctions U, V, $\delta\phi$, and $\delta\dot{\phi}$, and eigenfrequency ω. The eigenfunctions U and V fully prescribe the vector eigenfunction $\boldsymbol{s}_k(\boldsymbol{r})$. In component form, the set of four coupled first-order differential equations can be written:

$$\frac{\mathrm{d}}{\mathrm{d}r}\begin{bmatrix} U \\ \Delta P \\ \delta\phi \\ \psi \end{bmatrix} = \begin{bmatrix} -2/r+Dg_0 & D/\rho_0-1/\kappa_0 & D & 0 \\ \rho_0(-Dg_0^2+4g_0 r^{-1}+\omega^2) & -Dg_0 & \rho_0(-Dg_0+(l+1)/r) & -\rho_0 \\ -4\pi G\rho_0 & 0 & -(l+1)/r & 1 \\ 4\pi G\rho_0(Dg_0-(l+1)/r) & 4\pi GD & 4\pi G\rho_0 D & (l-1)/r \end{bmatrix}\begin{bmatrix} U \\ \Delta P \\ \delta\phi \\ \psi \end{bmatrix}, \tag{A 1}$$

where $g_0 = \dot{\phi}_0$ is the acceleration due to gravity, and where we have defined

$$\psi = \delta\dot{\phi} + (l+1)\,\delta\phi/r + 4\pi G\rho_0\,U, \tag{A 2}$$

$$D = l(l+1)/r^2\omega^2. \tag{A 3}$$

The eigenfunctions U, ΔP, $\delta\phi$, and ψ must satisfy two inner and two outer boundary conditions. The outer boundary conditions are

$$\psi = 0 \quad \text{at} \quad r = R_{\odot}, \tag{A 4}$$

$$\Delta P = 0 \quad \text{at} \quad r = R_{\odot}. \tag{A 5}$$

These boundary conditions were obtained by perturbing the boundary conditions in equations (47)–(49) and by using the perturbed Poisson equation. The inner boundary conditions can be obtained from eqs (13.7) and (13.8) of Unno *et al.* (1979). In our notation, these are given by

$$l\delta\phi - r\delta\dot{\phi} = 0 \quad \text{for} \quad r \sim 0, \tag{A 6}$$

$$r\omega^2 U - l(\Delta P/\rho_0 + g_0\,U + \delta\phi) = 0 \quad \text{for} \quad r \sim 0. \tag{A 7}$$

The system of equations (A 1) subject to the boundary conditions in equations (A 4)–(A 7) can be solved, for example, with the relaxation method described in Press *et al.* (1988) or with the more sophisticated technique described in Woodhouse (1988). Regardless of the method of solution, we also require $\dot{U}$, $\delta\dot{\phi}$, V, and $\dot{V}$. Once the eigenfunctions U, ΔP, $\delta\phi$, and ψ are calculated, expressions for $\dot{U}$ and $\delta\dot{\phi}$ follow directly from equation (A 1). The eigenfunction V is given by

$$V = (1/r\omega^2)\,(g_0\,U + \Delta P/\rho_0 + \delta\phi) \tag{A 8}$$

and $\dot{V}$ is obtained by direct differentiation of equation (A 8). Finally, we note that

$$\Delta P = -\kappa_0 \nabla\cdot\boldsymbol{s}$$
$$= -\kappa_0\,[\dot{U} + (2U - l(l+1)\,V)/r]. \tag{A 9}$$

Appendix B. Derivation of the equation of motion for the non-SNRNMAIS solar model

In this appendix we derive the equation of motion governing the seismic modes of the non-SNRNMAIS solar model. The lagrangian and eulerian seismic variations are specified in §B*a*. In §B*b*, the results of Lynden-Bell & Ostriker (1967) are extended by introducing aspherical perturbations to the elastic-gravitational variables.

(*a*) *Lagrangian and eulerian variations due to seismic motion*

The lagrangian change operator is defined by the operation

$$\Delta Q = Q(\boldsymbol{r} + \boldsymbol{s}(\boldsymbol{r}, t), t) - Q_0(\boldsymbol{r}, t), \tag{B 1}$$

where Q and Q_0 are the values of a scalar or vector quantity with and without oscillations, respectively, and $\boldsymbol{s}(\boldsymbol{r}, t)$ is the displacement undergone by a fluid element that would have been at $\boldsymbol{r}$ at time t in the absence of oscillations. This should be contrasted with the eulerian change of Q which is given by $\delta Q = Q(\boldsymbol{r}, t) - Q_0(\boldsymbol{r}, t)$. The lagrangian and eulerian operators are related to first order in the seismic displacement $\boldsymbol{s}(\boldsymbol{r}, t)$ by

$$\Delta = \delta + \boldsymbol{s}\cdot\nabla. \tag{B 2}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The eulerian and gradient operators commute; i.e.

$$[\nabla, \delta] = \mathbf{0}, \tag{B 3}$$

whereas the lagrangian and gradient operators do not.

Since by formal assumption (1) in §1*b*, we are assuming that convection is stationary relative to the corotating frame, the space and time dependence of the seismic motion separate in that frame and can be written

$$\boldsymbol{s}(\boldsymbol{r}, t) = \boldsymbol{s}(\boldsymbol{r})\, \mathrm{e}^{\mathrm{i}\omega t}, \tag{B 4}$$

analogous to equation (13). Consistent with formal assumption (2), the oscillation amplitudes are assumed to be small, so all perturbed quantities presented in the following are linearized in $\boldsymbol{s}(\boldsymbol{r})$.

If the equations of motion of a solar model do not include the advective part of the material time derivative (as with the SNRNMAIS model), the equations of motion governing seismic oscillations can be obtained with equal ease by taking either the lagrangian or eulerian variation of the unperturbed motion equation. However, if the reference state to which oscillations are added includes a velocity field, then the lagrangian rather than the eulerian variation should be taken since the lagrangian change operator Δ commutes with the material time derivative:

$$[\mathrm{D}/\mathrm{D}t, \Delta] = 0, \tag{B 5}$$

whereas the eulerian change operator δ does not.

We require the seismic lagrangian perturbations $\Delta\boldsymbol{r}$, $\Delta\boldsymbol{v}$, $\Delta\rho$, ΔP, and $\Delta\phi$ derived by Lynden-Bell & Ostriker (1967):

$$\Delta\boldsymbol{r} = \boldsymbol{s}, \tag{B 6}$$

$$\Delta\boldsymbol{v} = \frac{\mathrm{D}(\boldsymbol{r}+\boldsymbol{s})}{\mathrm{D}t} - \frac{\mathrm{D}\boldsymbol{r}}{\mathrm{D}t} = \frac{\mathrm{D}\boldsymbol{s}}{\mathrm{D}t}, \tag{B 7}$$

$$\Delta\rho = \delta\rho + \boldsymbol{s}\cdot\nabla\rho_0 = -\rho_0 \nabla\cdot\boldsymbol{s}, \tag{B 8}$$

$$\Delta\phi = \delta\phi + \boldsymbol{s}\cdot\nabla\phi_0, \tag{B 9}$$

$$\Delta P = \left(\frac{\partial P}{\partial \rho}\right)_S \Delta\rho + \left(\frac{\partial P}{\partial S}\right)_\rho \Delta S = \frac{P_0}{\rho_0}\left(\frac{\partial \ln P}{\partial \ln \rho}\right)_S \Delta\rho = \frac{P_0 \Gamma_1}{\rho_0}\Delta\rho = -\kappa_0 \nabla\cdot\boldsymbol{s}, \tag{B 10}$$

where $\Delta\rho$ was obtained by using the eulerian variation of the mass continuity equation ($\delta\rho = -\nabla\cdot(\rho\boldsymbol{s})$), and ΔP was obtained with the condition that the seismic motion is adiabatic; i.e. $\Delta S = 0$. The eulerian perturbation in the gravitational potential $\delta\phi$ can be obtained by solving the perturbed Poisson equation:

$$\nabla^2 \delta\phi = 4\pi G \delta\rho. \tag{B 11}$$

The eulerian seismic perturbation of the solar potential function Φ is given by

$$\delta\Phi = \delta\phi, \tag{B 12}$$

and the structural perturbation is

$$\begin{aligned} \delta\Phi_0 &= \delta\phi_0 + \delta\psi_c, \\ &= \delta\phi_0 + \psi_c, \end{aligned} \tag{B 13}$$

since $\psi_c = 0$ in the SNRNMAIS solar model.

(b) *The equation of motion*

To derive the equation of motion, we (1) take the lagrangian variation of the momentum equation (equation (41)), (2) assume the flow is steady state and signify this by replacing $\boldsymbol{v}$ with $\boldsymbol{u}_0$, (3) use equations (B 5)–(B 7) to simplify the inertial terms, and (4) introduce the modal ansatz given in equation (B 4). These operations produce an equation equivalent to eq. (28) in Lynden-Bell & Ostriker (1967):

$$\rho_0(r)\,[-\omega^2\boldsymbol{s}+2\mathrm{i}\omega(\boldsymbol{u}_0\cdot\nabla)\,\boldsymbol{s}+2\mathrm{i}\omega\boldsymbol{\Omega}\times\boldsymbol{s}+2\mathrm{i}\omega\boldsymbol{\Omega}\times[(\boldsymbol{u}_0\cdot\nabla)\,\boldsymbol{s}]+(\boldsymbol{u}_0\cdot\nabla)\,(\boldsymbol{u}_0\cdot\nabla)\,\boldsymbol{s}] = \mathscr{L}_0(\boldsymbol{s}), \tag{B 14}$$

where

$$\mathscr{L}_0(\boldsymbol{s}) = (\Delta\rho/\rho_0)\,\nabla P_0-\Delta(\nabla P)-\rho_0\,\Delta(\nabla\Phi). \tag{B 15}$$

Consistent with formal assumption (3) in §1*b*, we drop terms that depend quadratically on $\boldsymbol{u}_0$ and we drop terms of the order $(u_0\,\Omega)$ since these will be quite small for the Sun. Thus, equation (B 14) becomes

$$\rho_0(r)\,[-\omega^2\boldsymbol{s}+2\mathrm{i}\omega(\boldsymbol{u}_0\cdot\nabla)\,\boldsymbol{s}+2\mathrm{i}\omega\boldsymbol{\Omega}\times\boldsymbol{s}] = \mathscr{L}_0(\boldsymbol{s}). \tag{B 16}$$

We now seek to transform the operator $\mathscr{L}_0$ into a form equivalent but slightly different from the corresponding operator in Lynden-Bell & Ostriker (1967); the new form proves more convenient for the calculation of the general matrix elements in §6. The first term in $\mathscr{L}_0$ can be rewritten by using equations (45) and (B 8):

$$(\Delta\rho/\rho_0)\,\nabla P_0 = (\nabla\cdot\boldsymbol{s})\,\rho_0\,\nabla\Phi_0. \tag{B 17}$$

The second term in $\mathscr{L}_0$ can be rewritten:

$$\begin{aligned}\Delta(\nabla P) &= \delta(\nabla P)+\boldsymbol{s}\cdot\nabla(\nabla P_0)\\ &= \nabla\delta P+\boldsymbol{s}\cdot\nabla(\nabla P_0)\\ &= -\nabla(\kappa_0\,\nabla\cdot\boldsymbol{s})-\nabla(\boldsymbol{s}\cdot\nabla P_0)+\boldsymbol{s}\cdot\nabla(\nabla P_0)\\ &= -\nabla(\kappa_0\,\nabla\cdot\boldsymbol{s})+\nabla(\rho_0\,\boldsymbol{s}\cdot\nabla\Phi_0)-\boldsymbol{s}\cdot\nabla(\rho_0\,\nabla\Phi_0),\end{aligned} \tag{B 18}$$

where we have used equations (45), (B 2), (B 3) and (B 10). The third term in $\mathscr{L}_0$ can be rewritten:

$$\begin{aligned}\rho_0\,\Delta(\nabla\Phi) &= \rho_0\,\delta(\nabla\Phi)+\rho_0\,\boldsymbol{s}\cdot\nabla(\nabla\Phi_0)\\ &= \rho_0\,\nabla\delta\phi+\rho_0\,\boldsymbol{s}\cdot\nabla(\nabla\Phi_0),\end{aligned} \tag{B 19}$$

where we have used equations (B 2) and (B 3). Equations (B 15) and (B 17)–(B 19) together yield:

$$\mathscr{L}_0(\boldsymbol{s}) = (\nabla\cdot\boldsymbol{s})\,\rho_0\,\nabla\Phi_0+\nabla(\kappa_0\,\nabla\cdot\boldsymbol{s})-\nabla(\rho_0\,\boldsymbol{s}\cdot\nabla\,\Phi_0)+\boldsymbol{s}\cdot\nabla(\rho_0\,\nabla\Phi_0)-\rho_0\,\nabla\delta\phi -\rho_0\,\boldsymbol{s}\cdot\nabla(\nabla\Phi_0). \tag{B 20}$$

From equations (B 16) and (B 20) we obtain

$$-\rho_0\,\omega^2\boldsymbol{s}+\rho_0\,\boldsymbol{T}(\boldsymbol{s}) = \mathscr{L}_0(\boldsymbol{s}), \tag{B 21}$$

where we have defined

$$\boldsymbol{T}(\boldsymbol{s}) = \boldsymbol{B}(\boldsymbol{s})+\boldsymbol{C}(\boldsymbol{s}), \tag{B 22}$$

and

$$\boldsymbol{B}(\boldsymbol{s}) = 2\mathrm{i}\omega\boldsymbol{\Omega}\times\boldsymbol{s}, \tag{B 23}$$

$$\boldsymbol{C}(\boldsymbol{s}) = 2\mathrm{i}\omega\boldsymbol{u}_0\cdot\nabla\boldsymbol{s}. \tag{B 24}$$

It remains to introduce the aspherical perturbations in the elastic-gravitational variables κ_0, ρ_0, and ϕ_0 given by equations (29)–(31). We use the notation $\delta\mathscr{L}_0$ to

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

indicate the perturbation to the operator $\mathscr{L}_0$ due to the static perturbations in these variables. Taking the eulerian variation of the elastic-gravitational variables in $\mathscr{L}_0$, and retaining terms to first-order in these perturbations, we obtain

$$\delta\mathscr{L}_0(\boldsymbol{s}) = \nabla(\delta\kappa_0 \nabla\cdot\boldsymbol{s}) + (\nabla\cdot\boldsymbol{s} + \boldsymbol{s}\cdot\nabla)(\rho_0\nabla\delta\Phi_0 + \delta\rho_0\nabla\phi_0) - \nabla(\rho_0(\boldsymbol{s}\cdot\nabla\delta\Phi_0) + \delta\rho_0(\boldsymbol{s}\cdot\nabla\phi_0)) - \delta\rho_0\nabla\delta\phi(\boldsymbol{s}) - \delta\rho_0\,\boldsymbol{s}\cdot\nabla(\nabla\phi_0) - \rho_0\,\boldsymbol{s}\cdot\nabla(\nabla\delta\Phi_0). \quad \text{(B 25)}$$

Finally, from equation (B 21), the equation of motion in the presence of static aspherical structure and a steady-state velocity field in the corotating frame is given by

$$-(\rho_0+\delta\rho_0)\,\omega^2\boldsymbol{s} + \rho_0\,\boldsymbol{T}(\boldsymbol{s}) = \mathscr{L}_0(\boldsymbol{s}) + \delta\mathscr{L}_0(\boldsymbol{s}). \quad \text{(B 26)}$$

Appendix C. Converting differential operations on vector and tensor fields to algebraic operations in spherical coordinates

We show how a mathematical technique developed by Burridge (1969) and Phinney & Burridge (1973) can be applied to simplify the calculation of general matrix elements. The technique relies upon representing components of vectors and tensors in terms of generalized spherical harmonics $Y_l^{Nm}(\theta,\phi)$ (see equation (C 3)). The utility of the generalized spherical harmonics is that they transform differential operations acting on vector and tensor fields into algebraic operations. Furthermore, the integration of products of generalized spherical harmonics are easily expressed in terms of Wigner $3j$ symbols. Burridge (1969) defines a set of canonical coordinates in which the unit basis vectors of the coordinate system are the spherical unit vectors $\boldsymbol{e}_+ = \frac{-1}{\sqrt{2}}(\boldsymbol{e}_x + \mathrm{i}\boldsymbol{e}_y)$, $\boldsymbol{e}_0 = \boldsymbol{e}_z$, and $\boldsymbol{e}_- = \frac{-1}{\sqrt{2}}(\boldsymbol{e}_x - \mathrm{i}\boldsymbol{e}_y)$, where $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$ are the cartesian unit vectors. Phinney & Burridge (1973) provide the rules of vector-differential calculus appropriate to these canonical coordinates.

In these canonical coordinates the rules governing operations such as the inner product, the formation of gradients, and so forth are unfamiliar. For this reason, we present an alternative, hybrid technique that takes advantage of the properties of the generalized spherical harmonics, but at the same times preserves the familiar rules of vector-differential calculus. This is accomplished by retaining the basis vectors $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\varphi}}$, and by using the generalized spherical harmonics to express the components of vectors and tensors.

The hybrid technique may be summarized as follows. (1) Scalar functions (e.g. the elastic-gravitational perturbations $\delta\rho_0$, $\delta\kappa_0$, and $\delta\phi_0$) should be expanded in terms of the spherical harmonics in equation (C 4) (with index $N=0$). (2) The components of vector functions (e.g. the seismic displacement $\boldsymbol{s}_k$, the velocity field $\boldsymbol{u}_0$) should be expressed as in equations (C 15)–(C 20), and the components of second-order tensors should be expressed as in equations (C 22)–(C 30). (3) Vector-differential operations should be performed using equations (C 6) and (C 7), and equation (C 21). (4) The identities cited in §C*d* can be used to integrate products of generalized spherical harmonics over the unit sphere.

(*a*) *Generalized spherical harmonics*

The generalized spherical harmonics are related to the matrix elements of the finite rotation operator $D(\alpha, \beta, \gamma)$, where (α, β, γ) are the Euler angles. The matrix elements are given by

$$\mathscr{D}^l_{m'm}(\alpha,\beta,\gamma) = (lm'\,|D(\alpha,\beta,\gamma)|\,lm), \quad \text{(C 1)}$$

$$= \mathrm{e}^{\mathrm{i}m'\gamma} d^l_{m'm}(\beta)\,\mathrm{e}^{\mathrm{i}m\alpha}, \quad \text{(C 2)}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

(see eq. (4.1.11) in Edmonds 1960). The generalized spherical harmonics are defined in terms of the $\mathscr{D}^l_{m'm}$ functions by the relation

$$Y_l^{Nm}(\theta,\phi) = \mathscr{D}^l_{Nm}(\phi,\theta,0) = d^l_{Nm}(\theta)\,\mathrm{e}^{\mathrm{i}m\phi}, \tag{C 3}$$

where the function $d^l_{Nm}(\theta)$ is defined by eq. (4.1.23) of Edmonds (1960). The spherical harmonic $Y_l^m(\theta,\phi)$ (equation (3)) is a special case of the generalized spherical harmonics:

$$Y_l^m(\theta,\phi) = \gamma_l\, Y_l^{0m}(\theta,\phi), \tag{C 4}$$

where

$$\gamma_l = \surd((2l+1)/4\pi). \tag{C 5}$$

The generalized spherical harmonics satisfy numerous identities and recursion relationships (Gelfand & Shapiro 1956). In our application, the most useful of these are given by

$$\mathrm{d}Y_l^{Nm}(\theta,\phi)/\mathrm{d}\theta = \tfrac{1}{\sqrt{2}}(\Omega^l_N\, Y_l^{N-1,m}(\theta,\phi) - \Omega^l_{N+1}\, Y_l^{N+1,m}(\theta,\phi)), \tag{C 6}$$

$$((N\cos\theta - m)/\sin\theta)\, Y_l^{Nm}(\theta,\phi) = \tfrac{1}{\sqrt{2}}(\Omega^l_{N+1}\, Y_l^{N+1,m}(\theta,\phi) + \Omega^l_N\, Y_l^{N-1,m}(\theta,\phi)), \tag{C 7}$$

where

$$\Omega^l_N = \surd(\tfrac{1}{2}(l+N)\,(l-N+1)). \tag{C 8}$$

Higher derivatives of $Y_l^{Nm}(\theta,\phi)$ with respect to θ can be calculated by repeated application of equation (C 6); e.g.

$$\partial^2_\theta\, Y_l^{0m}(\theta,\phi) = \tfrac{1}{2}\Omega^l_0[\Omega^l_2(Y_l^{-2m}(\theta,\phi) + Y_l^{2m}(\theta,\phi)) - 2\Omega^l_0\, Y_l^{0m}(\theta,\phi)]. \tag{C 9}$$

(b) *Representing vector fields with generalized spherical harmonics*

By eq. (2.2) of Phinney & Burridge (1973), vectors expressed in terms of the unit basis vectors $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}})$ are transformed to the canonical representation by the relations

$$v^{-1} = \tfrac{1}{\sqrt{2}}[v_\theta + \mathrm{i}v_\phi], \quad v^0 = v_r, \quad v^1 = \tfrac{1}{\sqrt{2}}[-v_\theta + \mathrm{i}v_\phi], \tag{C 10}$$

and the $(-1,0,1)$ components are expanded in the form

$$v^{-1} = \sum_{lm} v^{-1}_{lm}\, Y_l^{-1m}, \quad v^0 = \sum_{lm} v^0_{lm}\, Y_l^{0m}, \quad v^1 = \sum_{lm} v^1_{lm}\, Y_l^{1m}. \tag{C 11}$$

The expansion coefficients $v^{\pm 1}_{lm}$ and v^0_{lm} may depend on the radial coordinate. The inverse transformations are given by

$$v_\theta = \tfrac{1}{\sqrt{2}}[v^{-1} - v^1], \quad v_\phi = -\mathrm{i}\tfrac{1}{\sqrt{2}}(v^{-1} + v^1), \quad v_r = v^0. \tag{C 12}$$

The appropriate representation of the vector $\boldsymbol{v}$ in terms of generalized spherical harmonics is obtained by applying, sequentially, equations (C 10)–(C 12). For example, by using equations (C 3), (C 38), (22), and the transformation rules in equations (C 10)–(C 12), the rotation vector $\boldsymbol{\Omega}$ can be written

$$\boldsymbol{\Omega} = \Omega Y_1^{00}(\theta,\phi)\,\hat{\boldsymbol{r}} + \tfrac{1}{\sqrt{2}}\Omega(Y_1^{-10}(\theta,\phi) - Y_1^{10}(\theta,\phi))\,\hat{\boldsymbol{\theta}}. \tag{C 13}$$

We have expressed the convective velocity field $\boldsymbol{u}_0$ and the seismic displacement $\boldsymbol{s}_k$ in terms of vector spherical harmonics. A general vector field can be written

$$\begin{aligned}\boldsymbol{v}(r,\theta,\phi) &= \sum_{s=0}^{\infty}\sum_{t=-s}^{s} v_1(r)\, Y_s^t\,\hat{\boldsymbol{r}} + v_2(r)\,\nabla_1\, Y_s^t - v_3(r)\,\hat{\boldsymbol{r}}\times\nabla_1\, Y_s^t \\ &= \sum_{s=0}^{\infty}\sum_{t=-s}^{s} v_1(r)\, Y_s^t\,\hat{\boldsymbol{r}} + \left(v_2(r)\,\partial_\theta\, Y_s^t + \frac{v_3(r)}{\sin\theta}\partial_\phi\, Y_s^t\right)\hat{\boldsymbol{\theta}} + \left(-v_3(r)\,\partial_\theta\, Y_s^t + \frac{v_2(r)}{\sin\theta}\partial_\phi\, Y_s^t\right)\hat{\boldsymbol{\phi}},\end{aligned} \tag{C 14}$$

where in general the radial expansion coefficients depend on t and s. Following a procedure similar to the derivation of equation (C 13), and by using equations (C 4)–(C 8) and equation (C 14), the components of $\boldsymbol{u}_0$ (equation (23)) can be written:

$$u_{0,r} = \gamma_s U_s^t(r)\, Y_s^{0t}, \tag{C 15}$$

$$u_{0,\theta} = \tfrac{1}{\sqrt{2}}\gamma_s \Omega_0^s\, V_s^t(r)\,(Y_s^{-1t} - Y_s^{1t}) - \tfrac{1}{\sqrt{2}}\mathrm{i}\gamma_s \Omega_0^s\, W_s^t(r)\,(Y_s^{1t} + Y_s^{-1t}), \tag{C 16}$$

$$u_{0,\phi} = -\tfrac{1}{\sqrt{2}}\mathrm{i}\gamma_s \Omega_0^s\, V_s^t(r)\,(Y_s^{1t} + Y_s^{-1t}) - \tfrac{1}{\sqrt{2}}\gamma_s \Omega_0^s\, W_s^t(r)\,(Y_s^{-1t} - Y_s^{1t}), \tag{C 17}$$

and the components of $\boldsymbol{s}_k$ (equation (1)) can be written:

$$s_{k,r} = \gamma_l U(r)\, Y_l^{0m}, \tag{C 18}$$

$$s_{k,\theta} = \tfrac{1}{\sqrt{2}}\gamma_l \Omega_0^l\, V(r)\,(Y_l^{-1m} - Y_l^{1m}), \tag{C 19}$$

$$s_{k,\phi} = -\tfrac{1}{\sqrt{2}}\mathrm{i}\gamma_l \Omega_0^l\, V(r)\,(Y_l^{1m} + Y_l^{-1m}). \tag{C 20}$$

(c) *A specific example: calculation of the advection kernel*

In this section, we calculate the integral

$$2\mathrm{i}\omega_{\mathrm{ref}} \int \rho_0\, \boldsymbol{s}_{k'}^* \cdot \boldsymbol{u}_0 \cdot \nabla \boldsymbol{s}_k\, \mathrm{d}^3 r$$

(see equation (84)) to provide an example of the application of the generalized spherical harmonic formalism. We first construct $\boldsymbol{T} = \nabla \boldsymbol{s}$. This requires expressions for the derivatives of the spherical unit vectors:

$$\frac{\partial \hat{\boldsymbol{r}}}{\partial \theta} = \hat{\boldsymbol{\theta}}, \quad \frac{\partial \hat{\boldsymbol{r}}}{\partial \phi} = \hat{\boldsymbol{\varphi}} \sin\theta, \quad \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta} = -\hat{\boldsymbol{r}}, \quad \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \phi} = \hat{\boldsymbol{\varphi}} \cos\theta, \quad \frac{\partial \hat{\boldsymbol{\varphi}}}{\partial \theta} = 0, \quad \frac{\partial \hat{\boldsymbol{\varphi}}}{\partial \phi} = -\hat{\boldsymbol{r}} \sin\theta - \hat{\boldsymbol{\theta}} \cos\theta. \tag{C 21}$$

Applying $\nabla = \hat{\boldsymbol{r}}\,\partial_r + r^{-1}\nabla_1$ to $\boldsymbol{s}_k$, and by using equations (C 18)–(C 21), we obtain

$$T_{rr} = \gamma_l \dot{U} Y_l^{0m}, \tag{C 22}$$

$$T_{r\theta} = \tfrac{1}{\sqrt{2}}\gamma_l \Omega_0^l\, \dot{V}(Y_l^{-1m} - Y_l^{1m}), \tag{C 23}$$

$$T_{r\phi} = -\tfrac{1}{\sqrt{2}}\mathrm{i}\gamma_l \Omega_0^l\, \dot{V}(Y_l^{1m} + Y_l^{-1m}), \tag{C 24}$$

$$T_{\theta r} = r^{-1}\gamma_l[U\,\partial_\theta Y_l^{0m} - \tfrac{1}{\sqrt{2}}\Omega_0^l V(Y_l^{-1m} - Y_l^{1m})], \tag{C 25}$$

$$T_{\theta\theta} = r^{-1}\gamma_l[U Y_l^{0m} + \tfrac{1}{\sqrt{2}}\Omega_0^l\, V\partial_\theta(Y_l^{-1m} - Y_l^{1m})], \tag{C 26}$$

$$T_{\theta\phi} = r^{-1}\gamma_l[-\tfrac{1}{\sqrt{2}}\Omega_0^l\, V\partial_\theta(Y_l^{1m} + Y_l^{-1m})], \tag{C 27}$$

$$T_{\phi r} = r^{-1}\gamma_l(\mathrm{i}m \operatorname{cosec}\theta\, U Y_l^{0m} + \tfrac{1}{\sqrt{2}}\mathrm{i}\Omega_0^l\, V(Y_l^{1m} + Y_l^{-1m})), \tag{C 28}$$

$$T_{\phi\theta} = r^{-1}\gamma_l(\mathrm{i}m \operatorname{cosec}\theta\,[\tfrac{1}{\sqrt{2}}\Omega_0^l\, V(Y_l^{-1m} - Y_l^{1m})] + \mathrm{i}\cot\theta[\tfrac{1}{\sqrt{2}}\Omega_0^l\, V(Y_l^{1m} + Y_l^{-1m})]), \tag{C 29}$$

$$T_{\phi\phi} = r^{-1}\gamma_l(U Y_l^{0m} + \cot\theta\,\tfrac{1}{\sqrt{2}}\Omega_0^l\, V(Y_l^{-1m} - Y_l^{1m}) + m \operatorname{cosec}\theta[\tfrac{1}{\sqrt{2}}\Omega_0^l\, V(Y_l^{1m} + Y_l^{-1m})]). \tag{C 30}$$

We next form, sequentially, the vector $\boldsymbol{u}_0 \cdot \boldsymbol{T}$ and the scalar $\boldsymbol{s}_{k'}^* \cdot (\boldsymbol{u}_0 \cdot \boldsymbol{T})$. The factors with (θ, ϕ) dependence in this expression can be reduced to products of generalized spherical harmonics by using, if necessary, equation (C 7). We then multiply the entire expression by $2\mathrm{i}\omega_{\mathrm{ref}}\rho_0$, and integrate over the volume of the solar model. The integrals over the generalized spherical harmonics can be performed by using the identities cited in §C*d*. Together, these operations yield the following result:

$$2\mathrm{i}\omega_{\mathrm{ref}} \int \rho_0\, \boldsymbol{s}_{k'}^* \cdot \boldsymbol{u}_0 \cdot \nabla \boldsymbol{s}_k\, \mathrm{d}^3 \boldsymbol{r} = 2\omega_{\mathrm{ref}}\, 4\pi \gamma_{l'} \gamma_l (-1)^{m'} \sum_{s=0}^{\infty} \sum_{t=-s}^{s} \gamma_s \begin{pmatrix} l' & s & l \\ -m' & t & m \end{pmatrix}$$
$$\times \left\{ \int_0^{R_\odot} \rho_0 [\mathrm{i} u_s^t(r)\, \tilde{R}_s(r) + \mathrm{i} v_s^t(r)\, \tilde{H}_s(r) + w_s^t(r)\, \tilde{T}_s(r)]\, r^2\, \mathrm{d}r \right\}, \tag{C 31}$$

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF

where the poloidal flow kernels $\tilde{R}_s(r)$ and $\tilde{H}_s(r)$, and toroidal flow kernel $\tilde{T}_s(r)$ are given by

$$\tilde{R}_s = U'\dot{U}B^{(0)+}_{l'sl} + V'\dot{V}B^{(1)+}_{l'sl}, \tag{C 32}$$

$$r\tilde{H}_s = [UU' - VU']B^{(1)+}_{sl'l} + (1+(-1)^{(l'+s+l)})\, V'V\Omega_0^{l'}\,\Omega_0^s\,\Omega_0^l\,\Omega_2^l\begin{pmatrix} l' & s & l \\ 1 & 1 & -2 \end{pmatrix}$$
$$+ [V'U - \Omega_0^l\,\Omega_0^l\,V'V]B^{(1)+}_{sll'}, \tag{C 33}$$

$$r\tilde{T}_s = [UU' - VU']B^{(1)-}_{sl'l} + [V'U - \Omega_0^l\,\Omega_0^l\,V'V]B^{(1)-}_{sll'}$$
$$-(1-(-1)^{(l'+s+l)})\, V'V\Omega_0^{l'}\,\Omega_0^s\,\Omega_0^l\,\Omega_2^l\begin{pmatrix} l' & s & l \\ 1 & 1 & -2 \end{pmatrix}, \tag{C 34}$$

for $-l \leqslant m \leqslant l$, $-l' \leqslant m' \leqslant l'$, and where the overdot indicates the radial derivative. The factors γ_l and Ω_N^l are given by equations (C 5) and (C 8). The $B^{(N)\pm}_{ijk}$ coefficients are defined in equation (C 44). In § D *b*, we show how the anelastic condition may be incorporated into this result.

(*d*) *Integration of generalized spherical harmonics on the unit sphere*

The integral of the product of three generalized spherical harmonics is given by

$$\int_0^{2\pi}\int_0^{\pi}[Y_{l'}^{N'm'}(\theta,\phi)]^*\,Y_{l''}^{N''m''}(\theta,\phi)\,Y_l^{Nm}(\theta,\phi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$= 4\pi(-1)^{(N'-m')}\begin{pmatrix} l' & l'' & l \\ -N' & N'' & N \end{pmatrix}\begin{pmatrix} l' & l'' & l \\ -m' & m'' & m \end{pmatrix}. \tag{C 35}$$

The Wigner 3*j* symbols which appear on the right hand side of equation (C 35) are defined by Edmonds (1960) in terms of the Clebsch–Gordan coefficients $(j_1\,m_1\,J_2\,m_2\,|\,J_1\,J_2 j_3 - m_3)$:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-m_3}(2j_3+1)^{-1}(j_1\,m_1\,J_2\,m_2\,|\,J_1\,J_2 j_3 - m_3). \tag{C 36}$$

The Clebsch–Gordan coefficients are defined in turn by eq. (3.6.11) of Edmonds (1960). The integral of the product of two generalized spherical harmonics can be obtained from equation (C 35) by noting that $Y_0^{00} = 1$.

There are situations in which it is necessary to integrate the product of four or more generalized spherical harmonics, e.g. the calculation of the general matrix elements for magnetic fields. By combining equation (C 35) with the direct product formula (presented below), it is possible to integrate the product of an arbitrary number of generalized spherical harmonics. The direct product formula is used to express the product of two generalized spherical harmonics in terms of a linear combination of generalized spherical harmonics, and can be applied repeatedly until one has only a product of three generalized spherical harmonics. The direct product formula is given by

$$Y_{l_1}^{N_1m_1}Y_{l_2}^{N_2m_2} = (-1)^{(m-N)}\sum_{j=|l_1-l_2|}^{|l_1+l_2|}(2j+1)\begin{pmatrix} l_1 & l_2 & j \\ N_1 & N_2 & -N \end{pmatrix}\begin{pmatrix} l_1 & l_2 & j \\ m_1 & m_2 & -m \end{pmatrix}Y_j^{Nm}, \tag{C 37}$$

where $N = N_1 + N_2$, $m = m_1 + m_2$, and be derived from eq. (4.3.2) of Edmonds (1960).

The calculation of general matrix elements requires that each factor with (θ, ϕ) dependence be expressed in terms of generalized spherical harmonics, e.g. the rotation vector $\boldsymbol{\Omega}$ in equation (22). The expression of simple low-order trigonometric functions in terms of generalized spherical harmonics requires analytical representations of the $d^l_{Nm}(\theta)$ functions. In the case of $l = 1$, we have from Edmonds (1960):

$$\begin{bmatrix} d^{(1)}_{-1-1} = \frac{1}{2}(1+\cos\theta) & d^{(1)}_{-10} = -\frac{1}{\sqrt{2}}\sin\theta & d^{(1)}_{-11} = \frac{1}{2}(1-\cos\theta) \\ d^{(1)}_{0-1} = \frac{1}{\sqrt{2}}\sin\theta & d^{(1)}_{00} = \cos\theta & d^{(1)}_{01} = -\frac{1}{\sqrt{2}}\sin\theta \\ d^{(1)}_{1-1} = \frac{1}{2}(1-\cos\theta) & d^{(1)}_{10} = \frac{1}{\sqrt{2}}\sin\theta & d^{(1)}_{11} = \frac{1}{2}(1+\cos\theta) \end{bmatrix}. \tag{C 38}$$

The properties of the Wigner $3j$ symbols we used to derive the general matrix elements include

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix}$$

$$= \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix} \tag{C 39}$$

$$= (-1)^{(j_1+j_2+j_3)} \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix}$$

$$= (-1)^{(j_1+j_2+j_3)} \begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix}$$

$$= (-1)^{(j_1+j_2+j_3)} \begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix} \tag{C 40}$$

$$= (-1)^{(j_1+j_2+j_3)} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} \tag{C 41}$$

$$= 0, \quad \text{if} \quad m_1 = m_2 = m_3 = 0 \quad \text{and} \quad j_1+j_2+j_3 \text{ is odd}, \tag{C 42}$$

$$\neq 0, \quad \text{only if} \quad m_1+m_2+m_3 = 0, |m_1| \leqslant j_1, |m_2| \leqslant j_2, |m_3| \leqslant j_3,$$

and

$$|j_1-j_2| \leqslant j_3, |j_2-j_3| \leqslant j_1, |j_3-j_1| \leqslant j_2. \tag{C 43}$$

(*e*) *The* $B^{(N)\pm}_{l'l''l}$ *coefficients*

The calculation of the reduced matrix elements (see (77) and (78)) generates frequently recurring terms, which, for notational simplicity, we have expressed in terms of the $B^{(N)\pm}_{ijk}$ coefficients defined by Woodhouse (1980):

$$B^{(N)\pm}_{l'l''l} = \tfrac{1}{2}(1 \pm (-1)^{(l'+l''+l)}) \left[\frac{(l'+N)!(l+N)!}{(l'-N)!(l-N)!}\right]^{\frac{1}{2}} (-1)^N \begin{pmatrix} l' & l'' & l \\ -N & 0 & N \end{pmatrix}. \tag{C 44}$$

There are several useful identities that the $B^{(N)\pm}_{ijk}$ coefficients satisfy (see Woodhouse 1980, eqs (A 43) and (A 46)):

$$B^{(1)+}_{l'sl} = \tfrac{1}{2}[l'(l'+1)+l(l+1)-s(s+1)]\,B^{(0)+}_{l'sl}, \tag{C 45}$$

$$B^{(1)-}_{l'sl} = \frac{1}{2}\left\{\frac{(\Sigma+2)(\Sigma+4)}{\Sigma+3}(\Sigma+1-2l)(\Sigma+1-2l')(\Sigma+1-2s)\right\}^{\frac{1}{2}} B^{(0)+}_{l'+1,\,s+1,\,l+1}, \tag{C 46}$$

where $\Sigma = l'+s+l$.

Appendix D. Calculation of the general matrix elements $B^{m'm}_{n'n,l'l}$ and $C^{m'm}_{n'n,l'l}$

(a) *The general matrix element* $B^{m'm}_{n'n,l'l}$

The general matrix element $B^{m'm}_{n'n,l'l}$ (equation (83)) governs Coriolis coupling and splitting. By using the representation of $\boldsymbol{\Omega}$ in equation (C 13), we obtain

$$B^{m'm}_{n'n,l'l} = \delta_{m'm}\,\delta_{l'l}\,2\omega_{\mathrm{ref}}\,m\Omega\int_0^{R_\odot}\rho_0(UV'+U'V+VV')\,r^2\,\mathrm{d}r. \qquad \text{(D 1)}$$

(b) *The general matrix element* $C^{m'm}_{n'n,l'l}$

We discuss the calculation of $C^{m'm}_{n'n,l'l}$ (equation (84)) in detail since the incorporation of the anelastic condition is complicated. The integral

$$2\mathrm{i}\omega_{\mathrm{ref}}\int\rho_0\,\boldsymbol{s}^*_{k'}\cdot(\boldsymbol{u}_0\cdot\nabla\boldsymbol{s}_k)\,\mathrm{d}^3\boldsymbol{r}$$

for a general flow field $\boldsymbol{u}_0$ (without the incorporation of the anelastic condition) was calculated in §C*c* (see (C 31)). The anelastic condition (equation (27)) constrains the expansion coefficients of the poloidal components of the velocity field $\boldsymbol{u}_0$. The incorporation of this constraint into the kernels $\tilde{R}_s(r)$ and $\tilde{H}_s(r)$ equations (C 32) and (C 33)) leads to a hermitian supermatrix. Our goal is to derive the kernels $R_s(r)$, $H_s(r)$, and $T_s(r)$, so that the substitutions,

$$\tilde{R}_s(r)\rightarrow R_s(r), \qquad \text{(D 2)}$$

$$\tilde{H}_s(r)\rightarrow H_s(r), \qquad \text{(D 3)}$$

$$\tilde{T}_s(r)\rightarrow T_s(r), \qquad \text{(D 4)}$$

yield the anelastic counterpart of equation (C 31). These kernels are given by equations (D 18)–(D 20).

The construction of these kernels will require two identities among Wigner $3j$ symbols:

Identity 1 $$-\tfrac{1}{2}\Omega^l_0\,\Omega^l_0\,B^{(1)+}_{sll'}+\tfrac{1}{2}(1+(-1)^{(l'+s+l)})\,\Omega^s_0\,\Omega^l_2\,\Omega^l_0\,\Omega^{l'}_0\begin{pmatrix} l' & s & l\\ 1 & 1 & -2\end{pmatrix}$$
$$=\tfrac{1}{4}[s(s+1)+l(l+1)-l'(l'+1)]\,B^{(1)+}_{l'sl}, \qquad \text{(D 5)}$$

Identity 2 $$\tfrac{1}{2}(1-(-1)^{(l'+s+l)})\,\Omega^s_0\,\Omega^l_2\,\Omega^l_0\,\Omega^{l'}_0\begin{pmatrix} l' & s & l\\ 1 & 1 & -2\end{pmatrix}-\tfrac{1}{2}\Omega^l_0\,\Omega^l_0\,B^{(1)-}_{sll'}$$
$$=\tfrac{1}{4}[l(l+1)+l'(l'+1)-s(s+1)]\,B^{(1)-}_{l'sl}. \qquad \text{(D 6)}$$

The proof of these identities requires the relationship

$$[(j_2-m_2)(j_2+m_2+1)(j_3+m_3)(j_3-m_3+1)]^{\frac{1}{2}}\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2+1 & m_3-1\end{pmatrix}$$
$$+[(j_2+m_2)(j_2-m_2+1)(j_3-m_3)(j_3+m_3+1)]^{\frac{1}{2}}\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2-1 & m_3+1\end{pmatrix}$$
$$=(j_1(j_1+1)-j_2(j_2+1)-j_3(j_3+1)-2m_2m_3)\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}, \qquad \text{(D 7)}$$

which can be derived from eq. (6.2.8) in Edmonds (1960). To prove Identity 1, we begin by setting $m_1 = -1$, $m_2 = 0$, $m_3 = 1$, $j_1 = l'$, $j_2 = s$, and $j_3 = l$ in equation (D 7) which yields

$$\Omega_0^s \Omega_0^l \begin{pmatrix} l' & s & l \\ 1 & -1 & 0 \end{pmatrix} + \Omega_1^s \Omega_2^l \begin{pmatrix} l' & s & l \\ 1 & 1 & -2 \end{pmatrix} = \tfrac{1}{2}[l'(l'+1) - l(l+1) - s(s+1)] \begin{pmatrix} l' & s & l \\ 1 & 0 & -1 \end{pmatrix}, \tag{D 8}$$

where we have used equations (C 8) and (C 41). Identity 1 can be obtained from equation (D 8) by rewriting the Wigner $3j$ symbols in terms of the $B_{ijk}^{(N)\pm}$ coefficients (equation (C 44)). To prove the second identity, we begin by using equations (C 44) and (D 8) to obtain

$$\tfrac{1}{2}\Omega^{l}{}_{0}\Omega_0^l B_{sll'}^{(1)-} + (1-(-1)^{(l'+s+l)})\,\Omega_0^s \Omega_2^l \Omega_0^l \Omega_0^{l'} \begin{pmatrix} l' & s & l \\ 1 & 1 & -2 \end{pmatrix}$$
$$= \tfrac{1}{4}[l'(l'+1) - l(l+1) - s(s+1)]\, B_{l'sl}^{(1)-}. \tag{D 9}$$

Adding the quantity $\frac{1}{2}l(l+1)B_{l'ls}^{(1)-}$ to both sides of equation (D 9) and recognizing that $B_{l'ls}^{(1)-} = B_{l'sl}^{(1)-}$ (which follows from the symmetry properties of (C 46)), we obtain the second identity.

Returning to equation (C 31), we note that the term in $\tilde{R}_s(r)$ may be written as a sum over two components, one symmetric, the other antisymmetric:

$$\int_0^{R_\odot} \rho_0(r)\,\mathrm{i}u_s^t(r)\,\tilde{R}_s(r)\,r^2\,\mathrm{d}r = \int_0^{R_\odot} r^2\rho_0(r)\,\mathrm{i}u_s^t(r)\,[\tfrac{1}{2}(U'\dot{U} + U\dot{U}')\,B_{l'sl}^{(0)+} + \tfrac{1}{2}(V'\dot{V} + V\dot{V}')\,B_{l'sl}^{(1)+}]\,\mathrm{d}r$$
$$+ \int_0^{R_\odot} r^2\rho_0(r)\,\mathrm{i}u_s^t(r)\,[\tfrac{1}{2}(U'\dot{U} - U\dot{U}')\,B_{l'sl}^{(0)+} + \tfrac{1}{2}(V'\dot{V} - V\dot{V}')\,B_{l'sl}^{(1)+}]\,\mathrm{d}r. \tag{D 10}$$

The symmetric quantity is antihermitian since it contains the factor i; the skew-symmetric term also contains the factor i and is therefore hermitian. An integration by parts of the symmetric term yields

$$\int_0^{R_\odot} \rho_0(r)\,\mathrm{i}u_s^t(r)\,[\tfrac{1}{2}(U'\dot{U} + U\dot{U}')\,B_{l'sl}^{(0)+} + \tfrac{1}{2}(V'\dot{V} + V\dot{V}')\,B_{l'sl}^{(1)+}]\,r^2\,\mathrm{d}r$$
$$= -\int_0^{R_\odot} \partial_r(r^2\rho_0\,\mathrm{i}u_s^t(r))\,[\tfrac{1}{2}U'UB_{l'sl}^{(0)+} + \tfrac{1}{2}V'VB_{l'sl}^{(1)+}]\,\mathrm{d}r. \tag{D 11}$$

Substituting the anelastic condition (27) into equation (D 11), and substituting the resulting expression into equation (D 10) yields

$$\int_0^{R_\odot} \rho_0\,\mathrm{i}u_s^t(r)\,\tilde{R}_s(r)\,r^2\,\mathrm{d}r = \int_0^{R_\odot} \rho_0(r)\,\mathrm{i}u_s^t(r)\,[\tfrac{1}{2}(U'\dot{U} - U\dot{U}')\,B_{l'sl}^{(0)+} + \tfrac{1}{2}(V'\dot{V} - V\dot{V}')\,B_{l'sl}^{(1)+}]\,r^2\,\mathrm{d}r$$
$$- \int_0^{R_\odot} \rho_0\,\mathrm{i}v_s^t(r)\,s(s+1)\,[\tfrac{1}{2}U'UB_{l'sl}^{(0)+} + \tfrac{1}{2}V'VB_{l'sl}^{(1)+}]\,r\,\mathrm{d}r. \tag{D 12}$$

Using equation (D 5), the kernel for horizontal poloidal flow may be rewritten

$$\int_0^{R_\odot} \rho_0\,\mathrm{i}v_s^t(r)\,\tilde{H}_s(r)\,r^2\,\mathrm{d}r = \int_0^{R_\odot} \rho_0\,\mathrm{i}v_s^t(r)\,[\tfrac{1}{2}[s(s+1) + l(l+1) - l'(l'+1)]\,VV'B_{l'sl}^{(1)+}$$
$$+ U'UB_{sl'l}^{(1)+} + V'UB_{l'ls}^{(1)+} - VU'B_{ll's}^{(1)+}]\,r\,\mathrm{d}r. \tag{D 13}$$

Using equation (D 6), the toroidal flow term in equation (C 31) may be replaced by

$$\int_0^{R_\odot} \rho_0 \, \mathrm{i} w_s^t(r)\, \tilde{T}_s(r)\, r^2\, \mathrm{d}r = \int_0^{R_\odot} r\rho_0\, w_s^t(r) \{(U'V - U'U)\, B_{sl'l}^{(1)-} + V'UB_{l'ls}^{(1)-} - \tfrac{1}{2}V'V[l(l+1) + l'(l'+1) - s(s+1)]\, B_{l'sl}^{(1)-}\}\, \mathrm{d}r. \quad \text{(D 14)}$$

Recognizing that $B_{l'sl}^{(1)-} = B_{sl'l}^{(1)-} = B_{l'ls}^{(1)-}$ (which follows from the symmetry properties of (C 46)), equation (D 14) reduces to

$$\int_0^{R_\odot} \rho_0 \, \mathrm{i} w_s^t(r)\, \tilde{T}_s(r)\, r^2\, \mathrm{d}r = \int_0^{R_\odot} r\rho_0\, w_s^t(r) \{U'V + V'U - U'U - \tfrac{1}{2}V'V[l(l+1) + l'(l'+1) - s(s+1)]\}\, B_{l'sl}^{(1)-}\, \mathrm{d}r. \quad \text{(D 15)}$$

We note that the right-hand sides of equations (D 12) and (D 13) contain, respectively, the factors $v_s^t\, U'UB_{l'sl}^{(0)+}$ and $v_s^t\, U'UB_{sl'l}^{(1)+}$. Since these terms sum together in equation (C 31), they can be combined by using the identity

$$B_{sl'l}^{(1)+} = \tfrac{1}{2}[s(s+1) + l(l+1) - l'(l'+1)]\, B_{l'sl}^{(0)+} \quad \text{(D 16)}$$

(equation (D 16) can be derived from (C 45)). Performing this operation, substituting the resulting expression and the remaining terms in equations (D 12), (D 13), and (D 15) into the right-hand side of equation (C 31), we obtain the final result:

$$C_{n'n,\,l'l}^{m'm} = 2\omega_{\mathrm{ref}}\, 4\pi \gamma_{l'} \gamma_l (-1)^{m'} \sum_{s=0}^{\infty} \sum_{t=-s}^{s} \gamma_s \begin{pmatrix} l' & s & l \\ -m' & t & m \end{pmatrix} \times \left\{ \int_0^{R_\odot} \rho_0 [\mathrm{i} u_s^t(r)\, R_s(r) + \mathrm{i} v_s^t(r)\, H_s(r) + w_s^t(r)\, T_s(r)]\, r^2\, \mathrm{d}r \right\}, \quad \text{(D 17)}$$

where the poloidal flow kernels $R_s(r)$, $H_s(r)$, and the toroidal flow kernel $T_s(r)$ are given by

$$R_s(r) = \tfrac{1}{2}(U'\dot{U} - \dot{U}'U)\, B_{l'sl}^{(0)+} + \tfrac{1}{2}(V'\dot{V} - \dot{V}'V)\, B_{l'sl}^{(1)+}, \quad \text{(D 18)}$$

$$rH_s(r) = \tfrac{1}{2}[l(l+1) - l'(l'+1)]\, [(U'U)\, B_{l'sl}^{(0)+} + (V'V)\, B_{l'sl}^{(1)+}] + V'UB_{l'ls}^{(1)+} - U'VB_{ll's}^{(1)+}, \quad \text{(D 19)}$$

$$rT_s(r) = \{U'V + V'U - U'U - \tfrac{1}{2}V'V[l(l+1) + l'(l'+1) - s(s+1)]\}\, B_{l'sl}^{(1)-}. \quad \text{(D 20)}$$

We note that the toroidal kernel is unconstrained by the anelastic condition and therefore $T_s(r) = \tilde{T}_s(r)$.

If desired, the functions $B_{l'sl}^{(1)+}, B_{ll's}^{(1)+}$, and $B_{l'sl}^{(1)-}$ may be reduced to expressions involving Wigner $3j$ symbols of the form

$$\begin{pmatrix} l' & s & l \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} l'+1 & s+1 & l+1 \\ 0 & 0 & 0 \end{pmatrix},$$

by making use of equations (C 45) and (C 46).

Appendix E. Calculation of the general matrix elements $K_{n'n,\,l'l}^{m'm}$, $R_{n'n,\,l'l}^{m'm}$ and $P_{n'n,\,l'l}^{m'm}$

The general matrix elements $K_{n'n,\,l'l}^{m'm}$, $R_{n'n,\,l'l}^{m'm}$ and $P_{n'n,\,l'l}^{m'm}$ (equations (86)–(88)) are complicated in that they contain derivatives of the elastic-gravitational perturbations $\delta\kappa_0$, $\delta\rho_0$, and $\delta\phi_0$. In the following we simplify the matrix elements by eliminating these derivatives whenever possible. In equations (E 18)–(E 20) we present the matrix elements in a form suitable for evaluation by application of the method described in Appendix C. The final result is given in equation (E 25).

(a) *The general matrix element* $K^{m'm}_{n'n,l'l}$

The general matrix element $K^{m'm}_{n'n,l'l}$ (equation (86)) governs modal interactions due to aspherical perturbations in the bulk modulus. Integrating $K^{m'm}_{n'n,l'l}$ by parts yields

$$K^{m'm}_{n'n,l'l} = -\int \delta\kappa_0 (\nabla\cdot\boldsymbol{s}^*_{k'})\,(\nabla\cdot\boldsymbol{s}_k)\,\mathrm{d}^3\boldsymbol{r}. \tag{E 1}$$

(b) *The general matrix element* $R^{m'm}_{n'n,l'l}$

The general matrix element $R^{m'm}_{n'n,l'l}$ (equation (87)) governs modal interactions due to aspherical perturbations in the density. By using the relations

$$\partial_r \phi_0 = g_0, \tag{E 2}$$

$$\partial^2_r \phi_0 = 4\pi G\rho_0 - 2g_0/r, \tag{E 3}$$

$$\boldsymbol{s}_k\cdot\nabla\hat{\boldsymbol{r}} = \boldsymbol{s}_k/r - (\hat{\boldsymbol{r}}/r)\,s^r_k \tag{E 4}$$

(where s^r_k denotes the radial component of $\boldsymbol{s}_k$ and g_0 is the gravitational acceleration), the integrands of the second and fourth terms in $R^{m'm}_{n'n,l'l}$ can be written

$$(\boldsymbol{s}^*_{k'}\cdot\nabla\phi_0)\,(\nabla\cdot\boldsymbol{s}_k) = g_0\, s^{r*}_{k'}\nabla\cdot\boldsymbol{s}_k, \tag{E 5}$$

$$\boldsymbol{s}^*_{k'}\cdot[\boldsymbol{s}_k\cdot\nabla(\nabla\phi_0)] = s^{r*}_{k'}\,s^r_k(4\pi G\rho_0 - 2g_0/r) + (g_0/r)\,(\boldsymbol{s}^*_{k'}\cdot\boldsymbol{s}_k - s^{r*}_{k'}\,s^r_k). \tag{E 6}$$

The fifth term in $R^{m'm}_{n'n,l'l}$ can be integrated by parts to yield

$$\int \boldsymbol{s}^*_{k'}\cdot\nabla(\delta\rho_0\,\boldsymbol{s}_k\cdot\nabla\phi_0)\,\mathrm{d}^3\boldsymbol{r} = -\int g_0\, s^r_k\,\nabla\cdot\boldsymbol{s}^*_{k'}\,\mathrm{d}^3\boldsymbol{r}. \tag{E 7}$$

After considerable manipulation, the last term in $R^{m'm}_{n'n,l'l}$ can be written

$$\int \boldsymbol{s}^*_{k'}\cdot[\boldsymbol{s}_k\cdot\nabla(\delta\rho_0\,\nabla\phi_0)]\,\mathrm{d}^3\boldsymbol{r} = \int g_0\,\delta\rho_0\left[-s^{r*}_{k'}\,\nabla\cdot\boldsymbol{s}_k - \boldsymbol{s}_k\cdot\nabla s^{r*}_{k'} + \frac{\boldsymbol{s}^*_{k'}\cdot\boldsymbol{s}_k}{r} - \frac{s^{r*}_{k'}\,s^r_k}{r}\right]\mathrm{d}^3\boldsymbol{r}. \tag{E 8}$$

With the above reductions we can rewrite $R^{m'm}_{n'n,l'l}$ in the form

$$R^{m'm}_{n'n,l'l} = \int \delta\rho_0\,[\omega^2_{\mathrm{ref}}\,\boldsymbol{s}^*_{k'}\cdot\boldsymbol{s}_k - \boldsymbol{s}^*_{k'}\cdot\nabla\delta\phi(\boldsymbol{s}_k) - 4\pi G\rho_0\,s^{r*}_{k'}\,s^r_k$$
$$+ g_0(s^r_k\nabla\cdot\boldsymbol{s}^*_{k'} - \boldsymbol{s}_k\cdot\nabla s^{r*}_{k'} + 2s^{r*}_{k'}\,s^r_k/r)]\,\mathrm{d}^3\boldsymbol{r}. \tag{E 9}$$

(c) *The general matrix element* $P^{m'm}_{n'n,l'l}$

The general matrix element $P^{m'm}_{n'n,l'l}$ (equation (88)) governs modal interactions due to aspherical perturbations in the gravitational potential. Integrating by parts the second term in equation (88) yields

$$\int \boldsymbol{s}^*_{k'}\cdot\nabla(\rho_0\,\boldsymbol{s}_k\cdot\nabla\delta\Phi_0)\,\mathrm{d}^3\boldsymbol{r} = \int \rho_0(\nabla\cdot\boldsymbol{s}^*_{k'})\,(\boldsymbol{s}_k\cdot\nabla\delta\Phi_0)\,\mathrm{d}^3\boldsymbol{r}. \tag{E 10}$$

The last two terms in $P^{m'm}_{n'n,l'l}$ can be written

$$\int \boldsymbol{s}^*_{k'}\cdot\{\boldsymbol{s}_k\cdot[\nabla(\rho_0\,\nabla\delta\Phi_0) - \rho_0\,\nabla(\nabla\delta\Phi_0)]\}\,\mathrm{d}^3\boldsymbol{r} = \int (\boldsymbol{s}^*_{k'}\cdot\nabla\rho_0)\,(\boldsymbol{s}_k\cdot\nabla\delta\Phi_0)\,\mathrm{d}^3\boldsymbol{r}. \tag{E 11}$$

Noting that

$$(\boldsymbol{s}_{k'}^{*}\cdot\nabla\rho_0)\,(\boldsymbol{s}_k\cdot\nabla\delta\Phi_0)=\nabla\cdot(\boldsymbol{s}_{k'}^{*}\,\rho_0\,\boldsymbol{s}_k\cdot\nabla\delta\Phi_0)-(\nabla\cdot\boldsymbol{s}_{k'}^{*})\,\rho_0\,\boldsymbol{s}_k\cdot\nabla\delta\Phi_0-\rho_0\,\boldsymbol{s}_{k'}^{*}\cdot\nabla(\boldsymbol{s}_k\cdot\nabla\delta\Phi_0), \qquad \text{(E 12)}$$

the right-hand side of equation (E 11) can be integrated by parts:

$$\int\boldsymbol{s}_{k'}^{*}\cdot\{(\nabla\rho_0)\,\boldsymbol{s}_k\cdot\nabla\delta\Phi_0\}\,\mathrm{d}^3\boldsymbol{r}=-\int\rho_0(\nabla\cdot\boldsymbol{s}_{k'}^{*})\,\boldsymbol{s}_k\cdot\nabla\delta\Phi_0\,\mathrm{d}^3\boldsymbol{r}-\int\rho_0\,\boldsymbol{s}_{k'}^{*}\cdot\nabla(\boldsymbol{s}_k\cdot\nabla\delta\Phi_0)\,\mathrm{d}^3\boldsymbol{r}. \qquad \text{(E 13)}$$

With these reductions, $P_{n'n,l'l}^{m'm}$ can be written

$$P_{n'n,l'l}^{m'm}=\int\rho_0\{\nabla\cdot\boldsymbol{s}_k(\boldsymbol{s}_{k'}^{*}\cdot\nabla\delta\Phi_0)-\boldsymbol{s}_{k'}^{*}\cdot\nabla(\boldsymbol{s}_k\cdot\nabla\delta\Phi_0)\}\,\mathrm{d}^3\boldsymbol{r}. \qquad \text{(E 14)}$$

(d) The symmetrization of $V_{n'n,l'l}^{m'm}$

To allow comparison of the foregoing results with the corresponding results of Woodhouse (1980), we express $V_{n'n,l'l}^{m'm}=K_{n'n,l'l}^{m'm}+P_{n'n,l'l}^{m'm}+R_{n'n,l'l}^{m'm}$ (see (85)) is as symmetric a form as possible. This can be accomplished by noting that $\mathscr{L}_0$ is a hermitian operator) i.e.

$$\int\boldsymbol{s}_{k'}^{*}\cdot\mathscr{L}_0(\boldsymbol{s}_k)\,\mathrm{d}^3\boldsymbol{r}=\int\boldsymbol{s}_k\cdot\mathscr{L}_0(\boldsymbol{s}_{k'}^{*})\,\mathrm{d}^3\boldsymbol{r}. \qquad \text{(E 15)}$$

From equation (E 15) it follows that

$$\int\boldsymbol{s}_{k'}^{*}\cdot\mathscr{L}_0(\boldsymbol{s}_k)\,\mathrm{d}^3\boldsymbol{r}=\frac{1}{2}\left[\int\boldsymbol{s}_{k'}^{*}\cdot\mathscr{L}_0(\boldsymbol{s}_k)\,\mathrm{d}^3\boldsymbol{r}+\int\boldsymbol{s}_k\cdot\mathscr{L}_0(\boldsymbol{s}_{k'}^{*})\,\mathrm{d}^3\boldsymbol{r}\right], \qquad \text{(E 16)}$$

$$\int\boldsymbol{s}_{k'}^{*}\cdot\delta\mathscr{L}_0(\boldsymbol{s}_k)\,\mathrm{d}^3\boldsymbol{r}=\frac{1}{2}\left[\int\boldsymbol{s}_{k'}^{*}\cdot\delta\mathscr{L}_0(\boldsymbol{s}_k)\,\mathrm{d}^3\boldsymbol{r}+\int\boldsymbol{s}_k\cdot\delta\mathscr{L}_0(\boldsymbol{s}_{k'}^{*})\,\mathrm{d}^3\boldsymbol{r}\right]. \qquad \text{(E 17)}$$

In §§Ea, b we evaluated $\int\boldsymbol{s}_{k'}^{*}\cdot\delta\mathscr{L}_0(\boldsymbol{s}_k)\,\mathrm{d}^3\boldsymbol{r}$. Thus the last term in the right-hand side of equation (E 17) can be obtained by switching the prime and unprimed variables in equations (E 1), (E 9), and (E 14). Performing this operation, and summing terms according to the prescription in equation (E 17), we obtain

$$K_{n'n,l'l}^{m'm}=-\int\delta\kappa_0(\nabla\cdot\boldsymbol{s}_{k'}^{*})\,(\nabla\cdot\boldsymbol{s}_k)\,\mathrm{d}^3\boldsymbol{r}, \qquad \text{(E 18)}$$

$$R_{n'n,l'l}^{m'm}=\int\delta\rho_0[\omega_{\mathrm{ref}}^2\,\boldsymbol{s}_{k'}^{*}\cdot\boldsymbol{s}_k-\boldsymbol{s}_{k'}^{*}\cdot\boldsymbol{\nabla}\delta\phi(\boldsymbol{s}_k)-\boldsymbol{s}_k\cdot\boldsymbol{\nabla}\delta\phi(\boldsymbol{s}_{k'}^{*})-4\pi G\rho_0\,s_{k'}^{r*}\,s_k^{r}$$
$$+\tfrac{1}{2}g_0(s_k^r\nabla\cdot\boldsymbol{s}_{k'}^{*}+s_{k'}^{r*}\nabla\cdot\boldsymbol{s}_k-\boldsymbol{s}_k\cdot\nabla s_{k'}^{r*}-\boldsymbol{s}_{k'}^{*}\cdot\nabla s_k^r+4s_{k'}^{r*}\,s_k^r/r)]\,\mathrm{d}^3\boldsymbol{r}, \qquad \text{(E 19)}$$

$$P_{n'n,l'l}^{m'm}=\frac{1}{2}\int\rho_0\,[(\nabla\cdot\boldsymbol{s}_k)\,\boldsymbol{s}_{k'}^{*}\cdot\nabla\delta\Phi_0+(\nabla\cdot\boldsymbol{s}_{k'}^{*})\,\boldsymbol{s}_k\cdot\nabla\delta\Phi_0$$
$$-\boldsymbol{s}_{k'}^{*}\cdot\nabla(\boldsymbol{s}_k\cdot\nabla\delta\Phi_0)-\boldsymbol{s}_k\cdot\nabla(\boldsymbol{s}_{k'}^{*}\cdot\nabla\delta\Phi_0)]\,\mathrm{d}^3\boldsymbol{r}. \qquad \text{(E 20)}$$

We have added the term $-\frac{1}{2}[\boldsymbol{s}_{k'}^{*}\cdot\nabla\delta\phi(\boldsymbol{s}_k)+\boldsymbol{s}_k\cdot\nabla\delta\phi(\boldsymbol{s}_{k'}^{*})]$ to $R_{n'n,l'l}^{m'm}$. Otherwise, the gravitational potential energy due to self-gravitation is not correctly modelled (see the discussion below eq. (2.2) in Luh (1974)). Finally, to emphasize the distinct contributions $\delta\phi_0$ and $\delta\psi_{\mathrm{c}}$ to $P_{n'n,l'l}^{m'm}$, we decompose it in the form

$$P_{n'n,l'l}^{m'm}=P_{n'n,l'l}^{(1)m'm}+P_{n'n,l'l}^{(2)m'm}, \qquad \text{(E 21)}$$

where

$$P^{(1)m'm}_{n'n,\,l'l} = \frac{1}{2}\int \rho_0[(\nabla\cdot \boldsymbol{s}_k)\,\boldsymbol{s}^*_{k'}\cdot\nabla\delta\phi_0 + (\nabla\cdot\boldsymbol{s}^*_{k'})\,\boldsymbol{s}_k\cdot\nabla\delta\phi_0 - \boldsymbol{s}^*_{k'}\cdot\nabla(\boldsymbol{s}_k\cdot\nabla\delta\phi_0) - \boldsymbol{s}_k\cdot\nabla(\boldsymbol{s}^*_{k'}\cdot\nabla\delta\phi_0)]\,\mathrm{d}^3\boldsymbol{r}, \quad \text{(E 22)}$$

$$P^{(2)m'm}_{n'n,\,l'l} = \frac{1}{2}\int \rho_0[(\nabla\cdot \boldsymbol{s}_k)\,\boldsymbol{s}^*_{k'}\cdot\nabla\psi_{\mathrm{c}} + (\nabla\cdot\boldsymbol{s}^*_{k'})\,\boldsymbol{s}_k\cdot\nabla\psi_{\mathrm{c}} - \boldsymbol{s}^*_{k'}\cdot\nabla(\boldsymbol{s}_k\cdot\nabla\psi_{\mathrm{c}}) - \boldsymbol{s}_k\cdot\nabla(\boldsymbol{s}^*_{k'}\cdot\nabla\psi_{\mathrm{c}})]\,\mathrm{d}^3\boldsymbol{r}. \quad \text{(E 23)}$$

We may compare the matrix elements obtained here with those of Woodhouse & Dahlen (1978) for the case of degenerate perturbation theory by setting $\boldsymbol{s}_k = \boldsymbol{s}_{k'}$. Doing so, we find that $B^{m'm}_{nn,\,ll}$ and $P^{(2)m'm}_{nn,\,ll}$, $R^{m'm}_{nn,\,ll}$ and $K^{m'm}_{nn,\,ll}$, and $P^{(1)m'm}_{nn,\,ll}$ agree, respectively, with eqn (68), (69), and (71) of Woodhouse & Dahlen (1978). Comparison of the matrix elements $B^{m'm}_{n'n,\,l'l}$, $P^{m'm}_{n'n,\,l'l}$, $R^{m'm}_{n'n,\,l'l}$, and $K^{m'm}_{n'n,\,l'l}$ with the first three integrals in eq. (A 1) of Woodhouse (1980) shows complete agreement for the case of quasi-degenerate perturbation theory.

(*e*) *The final form of* $V^{m'm}_{n'n,\,l'l}$

It remains to calculate $V^{m'm}_{n'n,\,l'l}$ using the representation of $\boldsymbol{s}_k$ in equation (1) and of $\delta\kappa_0$, $\delta\rho_0$, and $\delta\phi_0$ in equations (35)–(37). The integrals in equations (E 18), (E 19), and (E 20) can be calculated by using the method described in Appendix C. Alternatively, one may refer to Woodhouse (1980) where the same integrals have been calculated. Ultimately, one obtains integrals over $\delta\kappa_0$, $\delta\rho_0$, $\delta\phi_0$, and $\delta\dot{\phi}_0$. By using the relation

$$\frac{\partial^2\delta\phi^t_s(r)}{\partial^2 r} + \frac{2}{r}\frac{\partial\delta\phi^t_s(r)}{\partial r} - \frac{s(s+1)}{r^2}\delta\phi^t_s(r) = 4\pi G\delta\rho^t_s(r) \quad \text{(E 24)}$$

(which follows from Poisson's equation) and by performing several integrations by parts, it is possible to transform the integrals over $\delta\phi_0$ and $\delta\dot{\phi}_0$ into an integral over $\delta\rho_0$. We find $V^{m'm}_{n'n,\,l'l}$ can be written

$$V^{m'm}_{n'n,\,l'l} = E^m_l\int_0^{R_\odot} E(r)\,r^2\,\mathrm{d}r + \tfrac{2}{3}\Omega^2\delta_{l'l}\,\delta_{m'm}\left[\delta_{nn'} - l(l+1)\int_0^{R_\odot}\rho_0\,C(r)\,r^2\,\mathrm{d}r\right] + 4\pi\gamma_{l'}\gamma_l(-1)^{m'}\sum_{s=0}^{\infty}\sum_{t=-s}^{s}\gamma_s\begin{pmatrix} l' & s & l\\ -m' & t & m\end{pmatrix} \times\int_0^{R_\odot}[\delta\kappa^t_s(r)\,K_s(r) + \delta\rho^t_s(r)\,R^{(2)}_s(r)]\,r^2\,\mathrm{d}r, \quad \text{(E 25)}$$

where E^m_l, $E(r)$, $C(r)$, $K_s(r)$, and $R^{(2)}_s(r)$ are defined in §6*c*.

References

Backus, G. E. & Mulcahy, M. 1976*a* Moment tensors and other phenomenological descriptions of seismic sources. I. Continuous displacements. *Geophys. Jl R. astr. Soc.* **46**, 341–361.

Backus, G. E. & Mulcahy, M. 1976*b* Moment tensors and other phenomenological descriptions of seismic sources. II. Discontinuous displacements. *Geophys. Jl R. astr. Soc.* **47**, 301–329.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

van Ballegooijen, A. A. 1986 On the surface response of solar giant cells. *Ap. J.* **304**, 828–837.

Biot, M. A. 1965 *Mechanics of incremental deformations*. New York: Wiley.

Bogdan, T. 1989 On the interaction of the solar acoustic oscillations with convection. I. Scattering of sound by steady vorticity. *Ap. J.* **339**, 1132–1143.

Bray, R. J., Loughhead, R. E. & Durrant, C. J. 1984 *The solar granulation*. Cambridge University Press.

Brown, T. M. 1984 Solar *p*-mode eigenfrequencies are decreased by turbulent convection. *Science, Wash.* **226**, 687–689.

Brown, T. M. 1985 Solar rotation as a function of depth and latitude. *Nature, Lond.* **317**, 591–594.

Brown, T. M. & Gilman, P. A. 1984 Techniques for detecting giant cells using spatially resolved solar velocity data. *Ap. J.* **286**, 804–809.

Brown, T. M. & Morrow, C. A. 1987 Depth and latitude dependence of solar rotation. *Ap. J.* **314**, L21–L26.

Brown, T. M., Christensen-Dalsgaard, J., Dziembowski, W. A., Goode, P., Gough, D. O. & Morrow, C. A. 1989 Inferring the sun's internal angular velocity from observed *p*-mode frequency splittings. *Ap. J.* **343**, 526–546.

Brown, T. M. 1990 The source of solar high-frequency acoustic modes: theoretical expectations. *Ap. J.* **371**, 396–401.

Burridge, R. 1969 Spherically symmetric differential equations, the rotation group, and tensor spherical functions. *Proc. Camb. phil. Soc.* **65**, 157–175.

Chandrasekhar, S. & Roberts, P. H. 1963 The elipticity of a slowly rotating configuration. *Ap. J.* **138**, 801–808.

Christensen-Dalsgaard, J. & Fransden, S. 1983 Radiative transfer and solar oscillations. *Solar Phys.* **82**, 165–204.

Christensen-Dalsgaard, J., Gough, D. O. & Libbrecht, K. G. 1989 Seismology of solar oscillation line-widths. *Ap. J.* **341**, L103–L106.

Christensen-Dalsgaard, J., Schou, J. & Thompson, M. J. 1990 A comparison of methods for inverting helioseismic data. *Mon. Not. R. astr. Soc.* **242**, 353–369.

Cowling, T. G. & Newing, R. A. 1949 The oscillations of a rotating star. *Ap. J.* **109**, 149–158.

Cox, J. P. & Giuli, R. T. 1968 *Principles of stellar structure* New York: Gordon and Breach.

Dahlen, F. A. 1969 The normal modes of a rotating, elliptical Earth. II. Near resonance multiplet coupling. *Geophys. Jl R. Astr. Soc.* **18**, 397–436.

Dahlen, F. A. 1976 Reply. *J. geophys. Res.* **81**, 4951–4956.

Duvall, T. L. Jr & Harvey, J. W. 1984 Rotational frequency splitting of solar oscillations. *Nature, Lond.* **310**, 19–22.

Duvall, T. L. Jr, Harvey, J. W. & Pomerantz, M. A. 1986 Latitude and depth variation of solar rotation. *Nature, Lond.* **321**, 500–501.

Dziewonski, A. M. & Woodhouse, J. H. 1983 Studies of the seismic source using normal mode theory. In *Earthquakes: observations, theory and interpretation*, pp. 45–137. Bologna: Soc. Italiana di Fisica.

Edmonds, A. R. 1960 *Angular momentum in quantum mechanics*. Princeton University Press.

Gelfand, I. M. & Shapiro, Z. Y. 1956 Representations of the group of rotations in three-dimensional space and their applications. *Am. math. Soc. Transl.* **2**, 207–316.

Gilbert, F. 1971 The diagonal sum rule and averaged eigenfrequencies. *Geophys. Jl R. astr. Soc.* **23**, 119–123.

Gilman, P. A. 1987 The solar dynamo: observations and theories of solar convection, global circulation, and magnetic fields. In *Physics of the Sun* (ed. P. A. Sturrock), pp. 95–153. Reidel.

Gilman, P. A. & Glatzmaier, G. A. 1981 Compressible convection in a rotating spherical shell. I. Anelastic equations. *Ap. J. Suppl.* **45**, 335–349.

Gilman, P. A. & Miller, J. 1986 Nonlinear convection of a compressible fluid in a rotating spherical shell. *Ap. J. Suppl.* **61**, 585–608.

Glatzmaier, G. A. & Gilman, P. A. 1981 Compressible convection in a rotating spherical shell. II. A linear anelastic model. *Ap. J. Suppl.* **45**, 351–380.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Glatzmaier, G. A. & Gilman, P. A. 1982 Compressible convection in a rotating spherical shell. V. Induced differential rotation and meridional circulation. *Ap. J.* **256**, 316–330.

Glatzmaier, G. A. 1984 Numerical simulations of stellar convective dynamos. I. The model and method. *J. Comp. Phys.* **55**, 461–484.

Goldreich, P. & Keeley, D. A. 1977*a* Solar seismology. I. The stability of the solar *p*-modes. *Ap. J.* **211**, 934–942.

Goldreich, P. & Keeley, D. A. 1977*b* Solar seismology. II. The stochastic excitation of the solar *p*-modes by turbulent convection. *Ap. J.* **212**, 243–251.

Goldreich, P. & Kumar, P. 1988 The interaction of acoustic radiation with turbulence. *Ap. J.* **326**, 462–478.

Goldreich, P. & Kumar, P. 1990 Wave generation by turbulent convection. *Ap. J.* **363**, 694–704.

Gough, D. O. 1981 A new measure of the solar rotation. *Mon. Not. R. Astr. Soc.* **196**, 731–745.

Gough, D. O. 1982 Evidence for an oblique magnetic solar rotator. *Nature, Lond.* **298**, 350–354.

Gough, D. O., Moore, D. R., Spiegel, E. A. & Weiss, N. O. 1976 Convective instability in a compressible atmosphere. II. *Ap. J.* **206**, 536–542.

Gough, D. O. & Toomre, J. 1983 On the detection of subphotospheric convective velocities and temperature fluctuations. *Solar Phys.* **82**, 401–410.

Hansen, C. J., Cox, J. P. & Van Horn, H. M. 1977 The effects of differential rotation on the splitting of nonradial modes of stellar oscillation. *Ap. J.* **217**, 151–159.

Hart, J. E., Toomre, J. , Deane, A. E., Hurlburt, N. E., Glatzmaier, G. A., Fichtl, G. H., Leslie, F., Fowlis, W. W. & Gilman, P. A. 1986 Laboratory experiments on planetary and stellar convection performed on SPACE-LAB 3. *Science, Wash.* **234**, 61–64.

Hill, F., Toomre, J. & November, L. J. 1983 Variability in the power spectrum of solar five-minute oscillations. *Solar Phys.* **82**, 411–425.

Hill, F. 1988 Rings and trumpets – three dimensional power spectra of solar oscillations. *Ap. J.* **333**, 996–1013.

Hill, F. 1989 Solar oscillation ring diagrams and large-scale flows. *Ap. J.* **343**, L69–L72.

Howard, R. & LaBonte, B. J. 1980 The Sun is observed to be a torsional oscillator with a period of 11 years. *Ap. J. Lett.* **239**, L33–L66.

Kuhn, J. R., Libbrecht, K. G. & Dicke, R. H. 1988 The surface temperature of the Sun and changes in the solar constant. *Science, Wash.* **242**, 908.

Kumar, P. & Goldreich, P. 1989 Nonlinear interactions among solar acoustic modes. *Ap. J.* **342**, 558–575.

Kumar, P., Franklin, J. & Goldreich, P. 1988 Distribution functions for the time-average energies of stochastically excited solar *p*-modes. *Ap. J.* **328**, 879–887.

LaBonte, B. J., Howard, R. & Gilman, P. A. 1981 An improved search for large-scale convection cells in the solar atmosphere. *Ap. J.* **250**, 796–798.

Latour, J., Toomre, J. & Zahm, J.-P. 1981 Stellar convection theory. III. Dynamical coupling of the two convection zones in A-type stars by penetrative motions. *Ap. J.* **248**, 1081–1098.

Lavely, E. M. & Ritzwoller, M. H. 1992 Average effects of convection on helioseismic line-widths and frequencies. *Ap. J.* (Submitted.)

Ledoux, P. 1951 The nonradial oscillation of gaseous stars and the problem of *Beta Canis Majoris*. *Ap. J.* **114**, 373–384.

Libbrecht, K. G. 1986 Is there an unusual solar core? *Nature, Lond.* **319**, 753–755.

Libbrecht, K. G. & Kaufman, J. M. 1988 Frequencies of high-degree solar oscillations. *Ap. J.* **324**, 1172–1183.

Libbrecht, K. G. 1989 Solar *p*-mode frequency splittings. *Ap. J.* **336**, 1092–1097.

Luh, P. C. 1974 Normal modes of a rotating, self-gravitating, inhomogeneous Earth. *Geophys. Jl R. astr. Soc.* **38**, 187–224.

Lynden-Bell, D. & Ostriker, J. P. 1967 The stability of differentially rotating bodies. *Mon. Not. R. astr. Soc.* **136**, 293–310.

Phinney, R. A. & Burridge, R. 1973 Representation of the elastic-gravitational excitation of a spherical Earth model by generalized spherical harmonics. *Geophys. Jl R. astr. Soc.* **34**, 451–487.

Podsiadlowski, P. 1989 Ph.D. thesis, Massachusetts Institute of Technology.

Press, W. H., Flannery, B. P., Teukolsky, S. A. & Vetterling, W. T. 1988 *Numerical recipes*. Cambridge University Press.

Rhodes, E. J. Jr, Cacciani, A., Korzennik, S., Tomczyk, S., Ulrich, R. K. & Woodward, M. F. 1990 Depth and latitude dependence of the solar internal angular velocity. *Ap. J.* **351**, 687–700.

Ritzwoller, M. H. & Lavely, E. M. 1991 A unified approach to the Helioseismic forward and inverse problems of differential rotation. *Ap. J.* **369**, 557–566.

Schulten, K. & Gordon, R. G. 1975 Semiclassical approximations to $3j$- and $6j$-coefficients for quantum-mechanical coupling of angular momenta. *J. math. Phys.* **16**, 1971–1988.

Spruit, H., Nordlund, A. & Title, A. 1990 Solar convection. In *A. Rev. Astr. Astrophys.* **28**, 263–301.

Stein, R. F. & Nordlund, A. 1989 Topology of convection beneath the solar surface. *Ap. J.* **342**, L95–L98.

Stix, M. 1981 Theory of the solar cycle. *Solar Phys.* **74**, 79–101.

Tassoul, J.-L. 1978 *Theory of rotating stars*. Princeton University Press.

Thompson, M. J. 1990 A new inversion of solar rotational splitting data. *Solar Phys.* **125**, 1–12.

Title, A. M., Tarbell, T. D., Topka, K. P., Fergursson, S. H., Shine, R. A. & the SOUP TEAM 1989 Statistical properties of solar granulation derived from the SOUP instrument on SPACELAB 2. *Ap. J.* **336**, 475–494.

Unno, W., Osaki, Y., Ando, H. & Shibahasi, H. 1979 *Nonradial oscillations of stars*. University of Tokyo Press.

Woodhouse, J. H. 1976 On Rayleigh's principle. *Geophys. Jl R. astr. Soc.* **46**, 11–22.

Woodhouse, J. H. 1980 The coupling and attenuation of nearly resonant multiplets in the Earth's free oscillation spectrum. *Geophys. Jl R. astr. Soc.* **61**, 261–283.

Woodhouse, J. H. 1988 The calculation of the eigenfrequencies and eigenfunctions of the free oscillations of the Earth and the Sun. In *Seismological algorithms* (ed. D. J. Doornbos), pp. 321–370.

Woodhouse, J. H. & Dahlen, F. A. 1978 The effect of a general aspherical perturbation on the free oscillations of the Earth. *Geophys. Jl R. astr. Soc.* **53**, 335–354.

Woodhouse, J. H. & Girnius, T. P. 1982 Surface waves and free oscillations in a regionalized Earth model. *Geophys. Jl. R. astr. Sci.* **68**, 653–673.

Zare, R. 1989 *Angular momentum*. New York: John Wiley.

Received 30 November 1990; revised 11 October 1991; accepted 20 November 1991